Graduate Stat. Mech

HW # 8 - Solution

Dr. Gassem Alzoubi

① ideal Fermi gas in 2D at $T=0$

a) $g(\varepsilon) = \frac{(2s+1)}{h^2}\int d^2q\, d^2p\, \delta(\varepsilon-\varepsilon_p)$; $\varepsilon_p = \frac{p^2}{2m}$

$$= \frac{(2s+1)A}{h^2}\int p\,dp\,d\theta\, \delta(\varepsilon-\varepsilon_p) = \frac{(2s+1)A\,2\pi}{h^2}\int_0^\infty p\,dp\, \delta(\varepsilon-\varepsilon_p)$$

$$= \frac{(2s+1)\,2\pi m A}{h^2}\int_0^\infty d\varepsilon_p\, \delta(\varepsilon-\varepsilon_p) = \frac{(2s+1)\,2\pi m\, A}{h^2} = KA$$

$= \text{constant}$

b) $N = \int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon = KA\int_0^{\varepsilon_F} d\varepsilon = KA\varepsilon_F \Rightarrow \varepsilon_F = \frac{N}{A}\frac{1}{K} = \frac{n}{K}$

where $n = \frac{N}{A}$ particle density

$$\Rightarrow \varepsilon_F = \frac{n\,h^2}{(2s+1)\,2\pi m}$$

c) $E = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\,d\varepsilon\, f(\varepsilon)$; $f(\varepsilon) = 1$ for $\varepsilon < \varepsilon_F$

$$= KA\int_0^{\varepsilon_F} \varepsilon\, d\varepsilon = KA\,\frac{\varepsilon_F^2}{2}$$

d) $\Omega = \sum_k \Omega_k = \int_0^{\varepsilon_F} \Omega_k\, g(\varepsilon) f(\varepsilon)\,d\varepsilon$; $\Omega_k = -k_BT\ln(1+e^{\beta(\mu-\varepsilon_k)})$

$= -k_BT\,KA\int_0^{\varepsilon_F} d\varepsilon\,\ln(1+e^{\beta(\mu-\varepsilon)})$; at $T=0$, $\mu = \varepsilon_F$

integrate by parts

$u = \ln(1+e^{\beta(\varepsilon_F-\varepsilon)})$, $dv = d\varepsilon$

$v = \varepsilon$

$$du = \frac{-\beta\, e^{\beta(\varepsilon_F-\varepsilon)}}{1+e^{\beta(\varepsilon_F-\varepsilon)}}$$

$$\Omega = -k_B T K A \left[\underbrace{\varepsilon \ln\left(1 + e^{\beta(\varepsilon_F - \varepsilon)}\right) \Big|_0^{\varepsilon_F}}_{0} + \beta \int_0^{\varepsilon_F} \frac{\varepsilon \, d\varepsilon \, e^{\beta(\varepsilon_F - \varepsilon)}}{1 + e^{\beta(\varepsilon_F - \varepsilon)}} \right.$$

Now at $T = 0$, we have

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \varepsilon_F)} + 1} = \frac{e^{\beta(\varepsilon_F - \varepsilon)}}{1 + e^{\beta(\varepsilon_F - \varepsilon)}}$$

$$\Rightarrow \Omega = -k_B T K A \beta \int_0^{\varepsilon_F} \varepsilon \, d\varepsilon \underbrace{f(\varepsilon)}_{=1} = -K A \frac{\varepsilon_F^2}{2}$$

$$= -\frac{(2s+1) 2\pi m A}{h^2} \frac{\varepsilon_F^2}{2} = -\frac{(2s+1)\pi m A}{h^2} \varepsilon_F^2$$

e) $\Omega = -K A \dfrac{\varepsilon_F^2}{2} = -E \equiv -PA$

$\Rightarrow \quad PA = E \quad$ equation of state

② 3D system of ultra-relativistic electron gas at $T=0$

a) $g(\varepsilon) = \frac{(2s+1)}{h^3}\int d^3q\, d^3p\, \delta(\varepsilon-\varepsilon_p)$; $\varepsilon_p = cp$

$$= \frac{(2s+1)\,4\pi V}{h^3}\int_0^\infty p^2\, dp\, \delta(\varepsilon-\varepsilon_p) = \frac{(2s+1)\,4\pi V}{h^3 c^3}\int_0^\infty d\varepsilon_p\, \varepsilon_p^2\, \delta(\varepsilon-\varepsilon_p)$$

$$= \frac{(2s+1)\,4\pi V}{(hc)^3}\,\varepsilon^2 = KV\varepsilon^2$$; where $K = \frac{(2s+1)\,4\pi}{(hc)^3}$

b) $N = \int_0^{\varepsilon_F} g(\varepsilon)\, f(\varepsilon)\, d\varepsilon = KV\int_0^{\varepsilon_F}\varepsilon^2 d\varepsilon = KV\frac{\varepsilon_F^3}{3}$

$$\Rightarrow \frac{N}{V} = n = \frac{K\varepsilon_F^3}{3} \Rightarrow \varepsilon_F^3 = \frac{3n}{K} = \frac{3n\, h^3 c^3}{(2s+1)\,4\pi}$$

c) $E = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\, f(\varepsilon)\, d\varepsilon = KV\int_0^{\varepsilon_F}\varepsilon^3 d\varepsilon = KV\frac{\varepsilon_F^4}{4}$

d) $\Omega = \int g(\varepsilon)\, f(\varepsilon)\, \Omega_k\, d\varepsilon$; $\Omega_k = -k_B T \ln(1+e^{\beta(\varepsilon_F-\varepsilon)})$

$$= -KV k_B T \int_0^{\varepsilon_F} d\varepsilon\; \varepsilon^2 \ln(1+e^{\beta(\varepsilon_F-\varepsilon)})$$

integrate by parts

$$= -KV k_B T\left[\frac{1}{3}\varepsilon^3 \cancelto{0}{\ln(1+e^{\beta(\varepsilon_F-\varepsilon)})}\Big|_0^{\varepsilon_F} + \frac{1}{3}\beta\int_0^{\varepsilon_F}\frac{\varepsilon^3\, d\varepsilon}{1+e^{\beta(\varepsilon-\varepsilon_F)}}\right]$$

$$= -KV k_B T\left[\frac{1}{3}\beta\int_0^{\varepsilon_F}\varepsilon^3 f(\varepsilon)\, d\varepsilon\right]$$; $f(\varepsilon) = \frac{1}{1+e^{\beta(\varepsilon-\varepsilon_F)}}$

$$= -KV k_B T\,\frac{1}{3}\,\beta\,\frac{\varepsilon_F^4}{4} = -\frac{KV}{12}\,\varepsilon_F^4$$

$$\Rightarrow \therefore \Omega = -\frac{\kappa V}{12} \varepsilon_F^4$$

but we found that $\varepsilon_F^3 = \frac{3n}{\kappa} \Rightarrow \varepsilon_F = \left(\frac{3n}{\kappa}\right)^{1/3}$

$$\varepsilon_F^4 = \left(\frac{3n}{\kappa}\right)^{4/3}$$

$$\Rightarrow \Omega = -\frac{\kappa V}{12}\left(\frac{3n}{\kappa}\right)^{4/3}$$

$$= -\frac{\kappa V}{12} \frac{1}{\kappa^{4/3}} (3n)^{4/3}$$

$$= -\frac{3^{4/3}}{12} \kappa^{-1/3} V (n)^{4/3} \equiv -PV$$

$$\Rightarrow P = \frac{3^{4/3}}{12} \kappa^{-1/3} (n)^{4/3} = \frac{3^{4/3}}{12}\left(\frac{(2s+1)4\pi}{h^3c^3}\right)^{-1/3}\left(\frac{N}{V}\right)^{4/3}$$

$$= \frac{3^{4/3}}{12}\left(\frac{h^3c^3}{(2s+1)4\pi}\right)^{1/3}\left(\frac{N}{V}\right)^{4/3}$$

$$= \frac{3^1\,3^{1/3}}{12}\left(\frac{h^3c^3}{(2s+1)4\pi}\right)^{1/3}\left(\frac{N}{V}\right)^{4/3}$$

$$P = \frac{1}{4}\left(\frac{3h^3c^3}{(2s+1)4\pi}\right)^{1/3}\left(\frac{N}{V}\right)^{4/3}$$

(3) 3D system of ultra-relativistic electron gas at $T << T_F$

a) same as done in problem 2

$$g(\varepsilon) = K V \varepsilon^2 \quad , \quad K = \frac{(2s+1)4\pi}{(hc)^3}$$

b) $$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$= KV \int_0^\infty \frac{\varepsilon^2 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = KV\left[\int_0^\mu \varepsilon^2 d\varepsilon + \frac{\pi^2}{6}(k_BT)^2 F'(\mu)\right]$$

where $F(\varepsilon) = \varepsilon^2$
$F'(\varepsilon) = 2\varepsilon$
$F'(\mu) = 2\mu$

$$= KV\left[\frac{1}{3}\mu^3 + \frac{\pi^2}{6}(k_BT)^2 \cdot 2\mu\right]$$

$$= KV\left[\frac{1}{3}\mu^3 + \frac{\pi^2}{3}(k_BT)^2 \mu\right] \quad , \text{ but } \varepsilon_F^3 = \frac{3n}{K} = \frac{3N}{KV}$$

$$\Rightarrow KV = \frac{3N}{\varepsilon_F^3}$$

$$\Rightarrow \cancel{N} = \frac{3\cancel{N}}{\varepsilon_F^3}\left[\frac{1}{3}\mu^3 + \frac{\pi^2}{3}(k_BT)^2\mu\right]$$

$$\varepsilon_F^3 = \mu^3 + \pi^2(k_BT)^2\mu \quad , \text{ where } N(T=300\,K) \approx N(T=0\,K)$$

$$= \mu^3\left[1 + \pi^2\left(\frac{k_BT}{\mu}\right)^2\right]$$

Can be replaced by ε_F as $\mu(T=300\,K) \approx \mu(T=0) = \varepsilon_F$

$$= \mu^3\left[1 + \pi^2\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]$$

$$\Rightarrow \varepsilon_F = \mu\left[1 + \pi^2\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]^{1/3}$$

$$\Rightarrow \mu = \frac{\varepsilon_F}{\left[1 + \pi^2\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]^{1/3}} = \varepsilon_F\left(1 + \pi^2\left(\frac{k_BT}{\varepsilon_F}\right)^2\right)^{-1/3}$$

$$\approx \varepsilon_F\left(1 - \frac{\pi^2}{3}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right)$$

as $k_BT << \varepsilon_F$
$k_BT << k_BT_F$
i.e $T << T_F$

$$\therefore \mu = \varepsilon_F - \frac{\pi^2}{3}\frac{(k_BT)^2}{\varepsilon_F} \equiv \varepsilon_F + \delta\mu$$

$$\Rightarrow \delta\mu = -\frac{\pi^2}{3}\frac{(k_BT)^2}{\varepsilon_F}$$

c) $$E = \int_0^\infty g(\varepsilon)\,\varepsilon\, f(\varepsilon)\, d\varepsilon$$

$$= KV\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} \quad ; \quad F(\varepsilon) = \varepsilon^3$$

$$= KV\left[\frac{\mu^4}{4} + \frac{\pi^2}{2}(k_BT)^2\mu^2\right] \quad ; \text{but } \mu = \varepsilon_F + \delta\mu$$

$$= KV\left[\frac{(\varepsilon_F+\delta\mu)^4}{4} + \frac{\pi^2}{2}(k_BT)^2(\varepsilon_F+\delta\mu)^2\right]$$

$$= KV\left[\frac{\varepsilon_F^4}{4}\left(1+\frac{\delta\mu}{\varepsilon_F}\right)^4 + \frac{\pi^2}{2}(k_BT)^2\varepsilon_F^2\left(1+\frac{\delta\mu}{\varepsilon_F}\right)^2\right]$$

$$= KV\left[\frac{\varepsilon_F^4}{4}\left(1+4\frac{\delta\mu}{\varepsilon_F}\right) + \frac{\pi^2}{2}(k_BT)^2\varepsilon_F^2\left(1+\frac{2\delta\mu}{\varepsilon_F}\right)\right]$$

$$= KV\left[\frac{1}{4}(\varepsilon_F^4 + 4\varepsilon_F^3\delta\mu) + \frac{\pi^2}{2}(k_BT)^2(\varepsilon_F^2 + 2\varepsilon_F\delta\mu)\right]$$

$$= KV\left[\frac{1}{4}\varepsilon_F^4 + \varepsilon_F^3\delta\mu + \frac{\pi^2}{2}(k_BT)^2\varepsilon_F^2 + \pi^2(k_BT)^2\varepsilon_F\delta\mu\right]$$

Now using $\delta\mu = -\frac{\pi^2}{3}\frac{(k_BT)^2}{\varepsilon_F}$, we get

$$= KV\left[\frac{1}{4}\varepsilon_F^4 - \frac{\pi^2}{3}\varepsilon_F^2(k_BT)^2 + \frac{\pi^2}{2}(k_BT)^2\varepsilon_F^2 - \cancel{\frac{\pi^4}{3}(k_BT)^4}\right]$$

very small as $T \ll T_F$

$$= KV\left[\frac{1}{4}\varepsilon_F^4 + (k_BT)^2\varepsilon_F^2\,\pi^2\left(\frac{1}{2}-\frac{1}{3}\right)\right]$$

$$= KV\left[\frac{1}{4}\varepsilon_F^4 + \frac{\pi^2}{6}(k_BT)^2\varepsilon_F^2\right]$$

$$\therefore E = \kappa V \frac{\varepsilon_F^4}{4}\left[1 + \frac{2}{3}\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

$$\Rightarrow \frac{E}{V} = \frac{\kappa \varepsilon_F^4}{4}\left[1 + \frac{2}{3}\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

Now $C_V = \frac{\partial}{\partial T}(E/V)$

$$= \frac{\kappa \varepsilon_F}{4}\left[\frac{2}{3}\pi^2 \; 2\left(\frac{k_B T}{\varepsilon_F}\right)\right]$$

$$= \frac{1}{3}\frac{\kappa \pi^2 k_B}{\varepsilon_F} T \quad \propto T \qquad \text{Q.E.D}$$

④ a) $$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{z^{-1}e^{x}+1} = \frac{1}{\Gamma(n)} \int_0 x^{n-1}\left(\frac{ze^{-x}}{1+ze^{-x}}\right)dx$$

Now $$\frac{ze^{-x}}{1-(-ze^{-x})} = ze^{-x}\sum_{j=0}^{\infty}(-ze^{-x})^{j}$$

$$= ze^{-x}\sum_{j=1}^{\infty}(-ze^{-x})^{j-1} = ze^{-x}\sum_{j=1}^{\infty}(-1)^{j-1}(ze^{-x})^{j-1}$$

$$= \sum_{j=1}^{\infty}(-1)^{j-1}(ze^{-x})^{j}$$

$$\Rightarrow f_n(z) = \frac{1}{\Gamma(n)} \sum_{j=1}^{\infty}(-1)^{j-1} z^{j} \int_0^\infty x^{n-1} e^{-xj}\,dx \quad ; \quad \text{let } xj = y$$

$j\,dx = dy$

$x = \frac{y}{j}$

$x^{n-1} = \frac{y^{n-1}}{j^{n-1}}$

$$= \frac{1}{\Gamma(n)} \sum_{j=1}^{\infty}(-1)^{j-1} z^{j} \int_0^\infty \frac{y^{n-1}}{j^{n-1}}\frac{dy}{j}e^{-y}$$

$$= \frac{1}{\Gamma(n)} \sum_{j=1}^{\infty}(-1)^{j-1}\frac{z^{j}}{j^{n}} \underbrace{\int_0^\infty dy\, y^{n-1}e^{-y}}_{\Gamma(n)}$$

$$= \sum_{j=1}^{\infty}(-1)^{j-1}\frac{z^{j}}{j^{n}}$$

Now $$f_n(1) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j^{n}}$$

$$= \sum_{j\ odd}\frac{1}{j^n} - \sum_{j\ even}\frac{1}{j^n}$$

$$= \sum_{j\ odd}\frac{1}{j^n} + \sum_{j\ even}\frac{1}{j^n} - 2\sum_{j\ even}\frac{1}{j^n}$$

$$f_n(1) = \underbrace{\sum_{odd} \frac{1}{j^n} + \sum_{even} \frac{1}{j^n}}_{\downarrow} - 2 \sum_{even} \frac{1}{j^n}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^n} - 2 \sum_{j=1}^{\infty} \frac{1}{(2j)^n}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^n} - \frac{2}{2^n} \sum_{j=1}^{\infty} \frac{1}{j^n}$$

$$= \left(1 - \frac{1}{2^{n-1}}\right) \sum_{j=1}^{\infty} \frac{1}{j^n} = \left(1 - \frac{1}{2^{n-1}}\right) \zeta(n)$$

when $\zeta(n) = \sum_{j=1}^{\infty} \frac{1}{j^n}$

b) $$\int_0^{\infty} \frac{x^n\,dx}{e^x+1} = \int_0^{\infty} \frac{x^n e^{-x}}{\underbrace{1+e^{-x}}_{\to (1-(-e^{-x}))}} dx = \int_0^{\infty} x^n e^{-x} \left(\sum_{k=0}^{\infty} (-e^{-x})^k\right) dx$$

$$= \int_0^{\infty} x^n e^{-x} \sum_{k=0}^{\infty} (-1)^k e^{-kx}\, dx = \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} x^n e^{-x(k+1)}\, dx$$

let $y = x(k+1)$, $dy = dx\,(k+1)$,

$$= \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} \frac{y^n}{(k+1)^n} \frac{dy}{(k+1)} e^{-y} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{n+1}} \underbrace{\int_0^{\infty} y^n e^{-y} dy}_{\Gamma(n+1) = n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{n+1}} \Gamma(n+1)$$

$$= \Gamma(n+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{n+1}}$$

$$\therefore \int_0^\infty \frac{x^n\,dx}{e^x+1} = \Gamma(n+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{n+1}}$$

$$= \Gamma(n+1)\left[\sum_{k\text{ even}}\frac{1}{(k+1)^{n+1}} - \sum_{k\text{ odd}}\frac{1}{(k+1)^{n+1}}\right]$$

$$= \Gamma(n+1)\left[\underbrace{\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{n+1}}}_{\mathrm{I}} - \underbrace{\sum_{k=0}^{\infty}\frac{1}{(2k+2)^{n+1}}}_{\mathrm{II}}\right]$$

I: $$\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n} = \underbrace{\sum_{k=0}\frac{1}{(2k+1)^n}}_{\text{odd}} + \underbrace{\sum_{k=0}\frac{1}{(2k+2)^n}}_{\text{even}}$$

$$= \sum_{k=0}\frac{1}{(2k+1)^n} + \frac{1}{2^n}\sum_{k=0}\frac{1}{(k+1)^n}$$

$$= \sum_{k=0}\frac{1}{(2k+1)^n} + \frac{1}{2^n}\underbrace{\sum_{k=1}\frac{1}{k^n}}_{\zeta(n)}$$

$$\Rightarrow \quad \zeta(n) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^n} + \frac{1}{2^n}\zeta(n)$$

$$\Rightarrow \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^n} = \left(1-\frac{1}{2^n}\right)\zeta(n)$$

let $n \to n+1$

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{n+1}} = \left(1-\frac{1}{2^{n+1}}\right)\zeta(n+1)$$

$$\mathrm{II} = \sum_{k=0}^{\infty} \frac{1}{(2k+2)^{n+1}} = \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}}$$

$$= \frac{1}{2^{n+1}} \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} = \frac{1}{2^{n+1}} \zeta(n+1)$$

$$\therefore \int_0^{\infty} \frac{x^n dx}{e^x+1} = \Gamma(n+1) \left[\left(1 - \frac{1}{2^{n+1}}\right) \zeta(n+1) - \frac{1}{2^{n+1}} \zeta(n+1) \right]$$

$$= \Gamma(n+1) \left\{ \zeta(n+1) - \frac{1}{2^{n+1}} \zeta(n+1) - \frac{1}{2^{n+1}} \zeta(n+1) \right\}$$

$$= \Gamma(n+1) \left\{ \zeta(n+1) - \frac{2}{2^{n+1}} \zeta(n+1) \right\}$$

$$= \Gamma(n+1)\, \zeta(n+1) \left[1 - \frac{1}{2^n} \right]$$

Q.E.D.

(5) $C = C_{ele} + C_{pho}$, where $C_{ele} = \frac{\pi^2}{2} N k_B \frac{T}{T_F}$

and $C_{pho} = 234 N k_B \left(\frac{T}{T_D}\right)^3$

set $C_{ele} = C_{pho}$

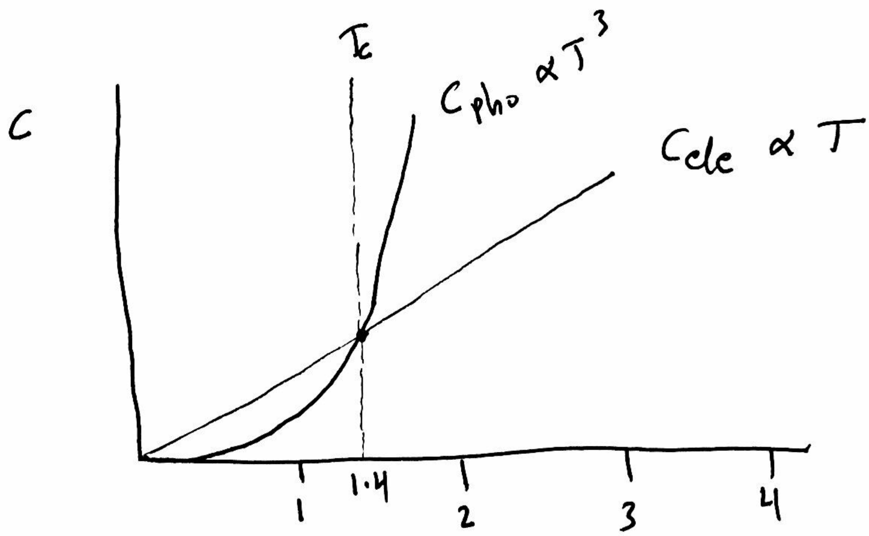
$$\Rightarrow \frac{\pi^2}{2} N k_B \frac{T_c}{T_F} = 234 N k_B \left(\frac{T_c}{T_D}\right)^3$$

$$\Rightarrow T_c^2 = \frac{\pi^2 T_D^3}{468 T_F}$$

$$\Rightarrow T_c = \left[\frac{\pi^2}{468} \frac{T_D^3}{T_F}\right]^{1/2}$$

for Na , $T_D \sim 150$ K and $T_F \sim 36{,}000$ K

$$\Rightarrow T_c \approx 1.4 \text{ K}$$



Note that at $T \ll T_c$, C_{ele} dominates C_{pho}

and at $T \gg T_c$, C_{pho} dominates C_{ele}