Graduate stat. Mech

HW # 7 - solution

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① $\varepsilon_p = c\,p$; c is speed of light ; $d\varepsilon_p = c\,dp$

a) $$g(\varepsilon) = \frac{(2s+1)}{h^3}\int d^3q\, d^3p\, \delta(\varepsilon - \varepsilon_p) = \frac{(2s+1)}{h^3}\, 4\pi V \int_0^\infty p^2 dp\, \delta(\varepsilon-\varepsilon_p)$$

$$= \frac{(2s+1)}{h^3}\frac{4\pi V}{c^3}\int_0^\infty \varepsilon_p^2\, d\varepsilon_p\, \delta(\varepsilon-\varepsilon_p) = \frac{(2s+1)}{h^3}\frac{4\pi V}{c^3}\varepsilon^2 = K V \varepsilon^2$$

where $K = \frac{(2s+1)}{h^3}\frac{4\pi}{c^3}$

b) $N = N_{ex} + N_0$; where $N_{ex} = \int_0^\infty d\varepsilon\, g(\varepsilon) f(\varepsilon)$; $f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} = \frac{1}{z^{-1}e^{\beta\varepsilon} - 1}$

$N_{ex} = KV\int_0^\infty \frac{\varepsilon^2 d\varepsilon}{z^{-1}e^{\beta\varepsilon} - 1}$; let $\beta\varepsilon = x$, $\beta\, d\varepsilon = dx$

$$= \frac{KV}{\beta^3}\int_0^\infty \frac{x^2 dx}{z^{-1}e^{x} - 1} = \frac{KV}{\beta^3}\int_0^\infty \frac{x^{3-1} dx}{z^{-1}e^{x}-1} = \frac{KV}{\beta^3}\, g_3(z)\, \underbrace{\Gamma(3)}_{2}$$

$$\boxed{N_{ex} = \frac{2KV}{\beta^3}\, g_3(z)} \quad \text{------ ------} \quad (1)$$

Now at $T = T_c$; $z = 1$ $\Rightarrow N_{ex} = 2KV k_B^3 T_c^3\, g_3(1)$

$\Rightarrow N = 2KV k_B^3 T_c^3 \zeta(3)$; where at $T \approx T_c$, $N_{ex} \approx N$, $g_3(1) = \zeta(3)$

$\Rightarrow \frac{N}{V} = 2 k_B^3 T_c^3 \zeta(3)\, K \Rightarrow T_c^3 = \frac{N}{V}\frac{1}{2k_B^3 \zeta(3) K}$ — (2)

$$\Rightarrow T_c^3 = \frac{N}{V} \frac{1}{2\,\zeta(3)} \frac{(hc)^3}{4\pi k_B^3 (2s+1)} \quad ; \quad \zeta(3) \approx 1.2$$

$$= \frac{N}{V} \frac{(hc)^3}{9.6\, k_B^3 \,\pi\, (2s+1)}$$

$$\Rightarrow T_c = \left[\frac{N}{V} \frac{(hc)^3}{9.6\, k_B^3\, \pi\, (2s+1)} \right]^{1/3} \quad --- \quad (3)$$

Now from equation (1), we have in general at any T

$$N_{ex} = \frac{2\pi V}{\beta^3} g_3(z) \quad , \text{ at } T \lesssim T_c \, , \; z=1$$

$$N_{ex} = \frac{2\pi V}{\beta^3} g_3(1) = 2\pi V k_B^3 T^3 \zeta(3) \times \frac{N}{N}$$

$$= \frac{N T^3}{N/2\pi V k_B^3 \zeta(3)} = \frac{N T^3}{T_c^3} = N \left(\frac{T}{T_c} \right)^3$$

c) $$E = \sum_k \varepsilon_k \langle n_k \rangle = \int_0^\infty \varepsilon\, g(\varepsilon)\, f(\varepsilon)\, d\varepsilon = \pi V \int_0^\infty \frac{\varepsilon^3 \, d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1}$$

$$= \pi V \int_0^\infty \frac{\varepsilon^3\, d\varepsilon}{z^{-1} e^{\beta\varepsilon} - 1} = \frac{\pi V}{\beta^4} \int_0^\infty \frac{x^3\, dx}{z^{-1} e^{x} - 1} = \frac{\pi V}{\beta^4} g_4(z) \underbrace{\Gamma(4)}_{3! = 6}$$

$$\Gamma(n) = (n-1)!$$

$$= \underbrace{\frac{\pi V}{\beta^3}}_{N_{ex}/2 g_3(z)} k_B T \; 6\, g_4(z) = \frac{N_{ex}}{2 g_3(z)} k_B T \; 6\, g_4(z)$$

from (1)

$$= 3 N_{ex} k_B T \frac{g_4(z)}{g_3(z)} \quad --- \quad (4)$$

i) at $T > T_c \Rightarrow N_{ex} \approx N$

$$\Rightarrow E = 3N k_B T \frac{g_4(z)}{g_3(z)} \quad \text{-----} \quad (5)$$

ii) at $T \lesssim T_c$; $N_{ex} = N\left(\frac{T}{T_c}\right)^3$ and $z=1$

from (4) $$E = 3N\left(\frac{T}{T_c}\right)^3 k_B T \frac{g_4(1)}{g_3(1)} = 3N\left(\frac{T}{T_c}\right)^3 k_B T \frac{\zeta(4)}{\zeta(3)}$$

$$= 3N k_B T_c \left(\frac{T}{T_c}\right)^4 \times \frac{1.0823}{1.2021}$$

$$\simeq 2.7 N k_B T_c \left(\frac{T}{T_c}\right)^4 \quad \text{-----} \quad (6)$$

d) $\Omega = \int g(\varepsilon)\, \Omega_k \, d\varepsilon$; $\Omega_k = k_B T \ln(1 - e^{\beta(\mu - \varepsilon_k)})$

$$= \int_0^\infty \pi V \varepsilon^2 k_B T \ln(1 - e^{\beta(\mu-\varepsilon)}) \, d\varepsilon$$

$$= \pi V k_B T \int_0^\infty d\varepsilon \; \varepsilon^2 \ln(1 - e^{\beta(\mu-\varepsilon)})$$

integrate by parts

$u = \ln(1 - e^{\beta(\mu-\varepsilon)})$, $dv = \varepsilon^2 d\varepsilon$

$du = \dfrac{\beta e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}}$ $\quad v = \frac{1}{3}\varepsilon^3$

$\Rightarrow$

$$\Omega = \pi V k_B T \left[\frac{1}{3}\varepsilon^3 \ln(1 - e^{\beta(\mu-\varepsilon)}) \Big|_0^\infty - \frac{1}{3}\beta \int_0^\infty \frac{\varepsilon^3 d\varepsilon \; e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}} \right.$$

$$\Omega = -\frac{\pi V}{3}\int_0^\infty \frac{\varepsilon^3\, d\varepsilon}{e^{-\beta(\mu-\varepsilon)}-1} = -\frac{\pi V}{3}\int_0^\infty \frac{\varepsilon^3\, d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1} \quad ; \quad \begin{aligned} &\text{let } x = \beta\varepsilon \\ &dx = \beta\, d\varepsilon \end{aligned}$$

$$= -\frac{\pi V}{3\beta^4}\int_0^\infty \frac{x^3\, dx}{z^{-1}e^{x}-1} = -\frac{\pi V}{3\beta^4}\int_0^\infty \frac{x^{4-1}\, dx}{z^{-1}e^{x}-1} = -\frac{\pi V}{3\beta^4}\, g_4(z)\, \underbrace{\Gamma(4)}_{3!\,=\,6}$$

$$= -2\frac{\pi V}{\beta^4} g_4(z) = -\underbrace{\frac{2\pi V}{\beta^3}}_{N_{ex}/g_3(z)} k_B T\, g_4(z)$$

$$= -\frac{N_{ex}}{g_3(z)} k_B T\, g_4(z) = -N_{ex} k_B T \frac{g_4(z)}{g_3(z)} \quad \text{------} \quad (7)$$

i) for $T > T_c$, $N_{ex} \approx N$

$$\Rightarrow \Omega = -N k_B T \frac{g_4(z)}{g_3(z)} \equiv -PV \Rightarrow P = \frac{N}{V} k_B T \frac{g_4(z)}{g_3(z)} \quad \text{---} \quad (8)$$

ii) for $T \lesssim T_c$, $N_{ex} = N\left(\frac{T}{T_c}\right)^3$ and $z = 1$

$$\Rightarrow \Omega = -N\left(\frac{T}{T_c}\right)^3 k_B T \frac{g_4(1)}{g_3(1)} = -N k_B T_c \left(\frac{T}{T_c}\right)^4 \frac{\zeta(4)}{\zeta(3)}$$

$$\equiv -PV$$

$$\Rightarrow P = \frac{N}{V} k_B T_c \left(\frac{T}{T_c}\right)^4 \frac{\zeta(4)}{\zeta(3)} \quad \text{------} \quad (9)$$

e) $\Omega = E - TS - \mu N \Rightarrow TS = E - \Omega - \mu N$ ...... (10)

i) for $T > T_c$; $N_{ex} \approx N$ and $z = e^{\beta\mu} \Rightarrow \mu = k_B T \ln z$

from (10)

$$TS = 3 N k_B T \frac{g_4(z)}{g_3(z)} + N k_B T \frac{g_4(z)}{g_3(z)} - k_B T \ln z$$

$$\Rightarrow \frac{S}{N k_B} = 3 \frac{g_4(z)}{g_3(z)} + \frac{g_4(z)}{g_3(z)} - \ln z$$

$$= 4 \frac{g_4(z)}{g_3(z)} - \ln z \quad \text{------} \quad (11)$$

ii) for $T \lesssim T_c$; $N_{ex} = N\left(\frac{T}{T_c}\right)^3$ and $z = 1$ ($\mu = 0$)

from (10) $$TS = 3N\left(\frac{T}{T_c}\right)^3 k_B T \frac{g_4(1)}{g_3(1)} + N\left(\frac{T}{T_c}\right)^3 k_B T \frac{g_4(1)}{g_3(1)}$$

$$\Rightarrow \frac{S}{N k_B} = 3\left(\frac{T}{T_c}\right)^3 \frac{\zeta(4)}{\zeta(3)} + \left(\frac{T}{T_c}\right)^3 \frac{\zeta(4)}{\zeta(3)}$$

$$= 4\left(\frac{T}{T_c}\right)^3 \frac{\zeta(4)}{\zeta(3)}$$

Note that as $T \to 0$, $S \to 0$ as expected from Third law of thermodynamics

(2) $$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^{x}-1} = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1}\left(\frac{1}{z^{-1}e^{x}-1}\right) dx$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1}\left(\frac{ze^{-x}}{1-ze^{-x}}\right) dx = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} ze^{-x}\left(\frac{1}{1-ze^{-x}}\right) dx$$

Now using $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$, we have $\frac{1}{1-ze^{-x}} = \sum_{j=0}^{\infty} (ze^{-x})^j$

$$\Rightarrow g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} dx \sum_{j=0}^{\infty} (ze^{-x})^{j+1} = \sum_{j=0}^{\infty} z^{j+1} \int_0^\infty x^{n-1} e^{-x(j+1)} dx$$

let $y = x(j+1) \Rightarrow x = \frac{y}{j+1}$; $x^{n-1} = \frac{y^{n-1}}{(j+1)^{n-1}}$

$$\Rightarrow g_n(z) = \frac{1}{\Gamma(n)} \sum_{j=0}^{\infty} z^{j+1} \int_0 \frac{y^{n-1}}{(j+1)^{n-1}} \frac{dy}{(j+1)} e^{-y}$$

$$= \frac{1}{\Gamma(n)} \sum_{j=0}^{\infty} z^{j+1} \frac{1}{(j+1)^n} \underbrace{\int_0^\infty dy\, y^{n-1} e^{-y}}_{\Gamma(n)}$$

$$= \sum_{j=0}^{\infty} \frac{z^{j+1}}{(j+1)^n} = \sum_{j=1}^{\infty} \frac{z^j}{j^n}$$

Now $g_n(1) = \sum_{j=1}^{\infty} \frac{1}{j^n} = \zeta(n)$ by deffinition

b) for $z=1 \Rightarrow g_n(1) = \sum_{j=1}^{\infty} \frac{1}{j^n}$; where $n > 0$

$$\int_1^\infty \frac{1}{x^n} dx = \lim_{M\to\infty} \int_1^M x^{-n} dx$$

$\Downarrow$

(i) if $n=1 \Rightarrow \ln x \Big|_1^M = \ln M - \cancelto{0}{\ln 1} = \ln M \xrightarrow{M\to\infty} \infty$

diverges

ii) if $n \neq 1 \Rightarrow \left.\dfrac{x^{-n+1}}{-n+1}\right|_1^M = \dfrac{M^{1-n}}{1-n} - \dfrac{1^{1-n}}{1-n}$

$$= \underbrace{\frac{M^{1-n}}{1-n}} - \frac{1}{1-n}$$

This term determines the convergence of the series

if $1-n > 0$ i.e positive, the series diverges $(n<1)$

if $1-n < 0$ i.e negative, the series converges $(n>1)$

c) $\displaystyle\int_0^\infty \frac{x^n\,dx}{e^x-1} = \int_0^\infty x^n e^{-x}\left(\frac{1}{1-e^{-x}}\right)dx = \sum_{k=0}^{\infty}\int x^n e^{-x}\left(e^{-x}\right)^k dx$

$\displaystyle = \sum_{k=0}^{\infty}\int x^n e^{-x(k+1)}\,dx$; let $y = x(k+1)$; $x = \dfrac{y}{k+1}$

$$x^n = \frac{y^n}{(k+1)^n}$$

$$= \sum_{k=0}^{\infty}\int \frac{y^n}{(k+1)^n}\,\frac{dy}{(k+1)}\,e^{-y}$$

$$= \sum_{k=0}^{\infty}\frac{1}{(k+1)^{n+1}}\underbrace{\int_0^\infty dy\, y^n e^{-y}} \quad = \Gamma(n+1)$$

$$= \Gamma(n+1)\sum_{k=0}^{\infty}\frac{1}{(k+1)^{n+1}} = \Gamma(n+1)\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}$$

$\displaystyle = \Gamma(n+1)\,\zeta(n+1)$; when $\displaystyle \zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$

Q.E.D

③ a) $g(\varepsilon) = \frac{(2s+1)}{h^2} \int d^2q \, d^2p \, \delta(\varepsilon - \varepsilon_p)$; $\varepsilon_p = \frac{p^2}{2m}$, $2m \, d\varepsilon_p = 2p\,dp$

$$= \frac{(2s+1)\, 2\pi A}{h^2} \int_0^\infty p\,dp \, \delta(\varepsilon - \varepsilon_p)$$

$d^2p = p\,dp\,d\theta$

$0 < p < \infty$, $0 < \theta < 2\pi$

$$= \frac{(2s+1)\, 2\pi m A}{h^2} \int_0^\infty d\varepsilon_p \, \delta(\varepsilon - \varepsilon_p) = \frac{(2s+1)\, 2\pi m A}{h^2}$$

$= KA =$ constant , when $K = \frac{(2s+1)\, 2\pi m}{h^2}$

b) $N = N_{ex} + N_0$; $N_{ex} = \int_0^\infty d\varepsilon \, g(\varepsilon) f(\varepsilon)$; $f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$

$= \frac{1}{z^{-1} e^{\beta\varepsilon} - 1}$

$\Rightarrow N_{ex} = KA \int_0^\infty \frac{d\varepsilon}{z^{-1} e^{\beta\varepsilon} - 1}$; let $\beta\varepsilon = x$

$$= \frac{KA}{\beta} \int_0^\infty \frac{dx}{z^{-1} e^{x} - 1} = \frac{KA}{\beta} \int_0^\infty \frac{x^0 \, dx}{z^{-1} e^{x} - 1}$$

$$= \frac{KA}{\beta} \int_0^\infty \frac{x^{1-1} \, dx}{z^{-1} e^{x} - 1} = KA \, k_B T \, g_1(z) \, \Gamma(1) \quad ; \quad \Gamma(1) = 1$$

$$= KA \, k_B T \, g_1(z) \text{ ------}$$

for B.E condensation to occur, μ must be equal to zero, i.e $z = 1$. but we see that $g_1(1)$ diverges for $n = 1$, meaning as $z \to 1$, $N_{ex} \to \infty$ indicating that excited states capacity is unbounded and hence can hold all available particles without any need for condensation to the ground state. So there is No B.E condensation in 2D. similar result holds also for 1D system. Exception occurs at $T = 0$, where $N_{ex} = 0$. However the zero temp can not be reached. Hence B.E condensation can not be realized.

(4) a) $g(\varepsilon) = \frac{(2s+1)}{h} \int dq\, dp\; \delta(\varepsilon - \varepsilon_p)$; $\varepsilon_p = p^2/2m$

$2m\varepsilon_p = p^2$

$2m\, d\varepsilon_p = 2p\, dp$

$dp = \frac{m}{p} d\varepsilon_p$

$= \frac{m}{(2m\varepsilon_p)^{1/2}} d\varepsilon_p$

$= \frac{m^{1/2}}{\sqrt{2}} \varepsilon_p^{-1/2} d\varepsilon_p$

$$= \frac{(2s+1)\, m^{1/2} L}{\sqrt{2}\, h} \int_0^\infty \varepsilon_p^{-1/2} d\varepsilon_p\, \delta(\varepsilon - \varepsilon_p)$$

$$= \frac{(2s+1)\sqrt{m}\, L}{\sqrt{2}\; h} \varepsilon^{-1/2}$$

$$= K L \varepsilon^{-1/2} ;$$

when $K = \frac{(2s+1)\sqrt{m}}{\sqrt{2}\; h}$

b) $N = N_{ex} + N_0$;

$$N_{ex} = \int_0^\infty d\varepsilon\; g(\varepsilon) f(\varepsilon) = KL \int_0^\infty \frac{d\varepsilon\; \varepsilon^{-1/2}}{z^{-1} e^{\beta\varepsilon} - 1}$$; let $\beta\varepsilon = x$

$\beta\, d\varepsilon = dx$

$\beta^{-1/2} \varepsilon^{-1/2} = x^{-1/2}$

$\varepsilon^{-1/2} = \frac{x^{-1/2}}{\beta^{-1/2}}$

$$= \frac{KL}{\beta^{1/2}} \int_0^\infty \frac{x^{-1/2} dx}{z^{-1} e^x - 1} = \frac{KL}{\beta^{1/2}} \int_0^\infty \frac{x^{\frac{1}{2}-1} dx}{z^{-1} e^x - 1}$$

$$= KL (k_\beta T)^{1/2} g_{1/2}(z)$$

Now at $T \approx T_c$, $z = 1$. but $g_{1/2}(1)$ diverges and hence $N_{ex} \to \infty$, meaning the excited state can hold all particles without any need for B. E condensation into the ground state. So B. E condensation does not occur in 1D. exception occurs at $T = 0$. but this situation can not be relaized.

⑤ Debye model in 2D

a) $g(\varepsilon) = \frac{1}{h^2} \sum_p \int d^2q \, d^2p \, \delta(\varepsilon - \varepsilon_p)$; $\underbrace{\varepsilon_p = c_p p}_{\Downarrow} = c_p \hbar k$

$d\varepsilon_p = c_p \, dp$

phonons are massless particles with three states of polarizations $(1L + 2T)$

$$g(\varepsilon) = \frac{A}{h^2} \sum_p 2\pi \int_0^\infty p \, dp \, \delta(\varepsilon - \varepsilon_p)$$

$$= \frac{2\pi A}{h^2} \sum_p \int_0^\infty \frac{\varepsilon_p}{c_p} \frac{d\varepsilon_p}{c_p} \delta(\varepsilon - \varepsilon_p) = \frac{2\pi A}{h^2} \left[\frac{1}{c_l^2} + \frac{2}{c_t^2} \right] \varepsilon$$

$$= \frac{2\pi A}{h^2} \frac{1}{\bar{c}^2} \varepsilon \quad ; \text{ where } \frac{1}{\bar{c}^2} = \frac{1}{c_l^2} + \frac{2}{c_t^2}$$

$$= K \varepsilon \quad ; \quad K = \frac{2\pi A}{h^2 \bar{c}^2}$$

$$\therefore g(\varepsilon) = \begin{cases} K\varepsilon & , \varepsilon < \varepsilon_D \\ 0 & , \varepsilon > \varepsilon_D \end{cases} \quad ; \ \varepsilon_D : \text{ debye energy}$$

from normalization condition, we have

$$\int_0^{\varepsilon_D} g(\varepsilon) \, d\varepsilon = 3N \Rightarrow \frac{K \varepsilon_D^2}{2} = 3N \Rightarrow \varepsilon_D^2 = \frac{6N}{K}$$

Now $\varepsilon_D = \hbar \omega_D = k_B T_D \Rightarrow \omega_D = \frac{k_B}{\hbar} T_D$

b) $E = \int_0^\infty \varepsilon \, g(\varepsilon) f(\varepsilon) \, d\varepsilon$, $f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$; but $\mu = 0$ for phonons

$$= \int_0^{\varepsilon_D} \frac{K \varepsilon^2 \, d\varepsilon}{e^{\beta\varepsilon} - 1} \qquad = \frac{1}{e^{\beta\varepsilon} - 1}$$

let $\beta \varepsilon = \frac{\varepsilon}{k_B T} = x \Rightarrow \varepsilon = k_B T x \Rightarrow d\varepsilon = k_B T \, dx$

$$d\varepsilon = \frac{1}{\beta} dx$$

now if $\varepsilon = 0 \Rightarrow x = 0$ ~~and if~~

if $\varepsilon = \varepsilon_D \Rightarrow x = \frac{\varepsilon_D}{k_B T} = \frac{k_B T_D}{k_B T} = \frac{T_D}{T}$

$$\Rightarrow E = \frac{K}{\beta^3} \int_0^{T_D/T} \frac{x^2\, dx}{e^x - 1} \quad ; \text{ when again } x = \frac{\varepsilon}{k_B T} = \frac{\hbar\omega}{k_B T}$$

i) high T limit $(k_B T >> \hbar\omega)$

$e^x \simeq 1 + x$; x is small

$$E = \frac{K}{\beta^3} \int_0^{T_D/T} x\, dx = \frac{K}{2\beta^3} T_D^2 = \frac{K}{2} k_B^3 T^3 T_D^2 \quad ; \text{ but } \varepsilon_D^2 = \frac{6N}{K}$$

$$\Rightarrow K = \frac{6N}{\varepsilon_D^2} = \frac{6N}{k_B^2 T_D^2}$$

$$= \frac{1}{2} \frac{6N}{k_B^2 T_D^2} k_B^3 T^3 T_D^2 = 3 N k_B T$$

$\Rightarrow C_V = 3 N k_B$ as expected

ii) low T limit $(k_B T << \hbar\omega)$; $\frac{T_D}{T} \to \infty$

$$E = \frac{K}{\beta^3} \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{K}{\beta^3} \int_0^\infty \frac{x^{3-1} dx}{e^x - 1} = \frac{K}{\beta^3} g_3(1)\, \Gamma(3)$$

$$= K k_B^3 T^3 \underbrace{\zeta(3)}_{1.2} \underbrace{\Gamma(3)}_{2} = \frac{6N}{k_B^2 T_D^2} k_B^3 T^3 (1.2)(2)$$

$$= 14.4\, N k_B \frac{T^3}{T_D^2}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3 \times 14.4\, N k_B \left(\frac{T}{T_D}\right)^2 = 43.3\, N k_B \frac{T^2}{T_D^2}$$