Graduate Stat. Mech

HW #6 – solution

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① Pathria 6.1

$$\Omega = \sum_k \Omega_k = \sum_k \eta k_B T \ln\left(1 - \eta e^{\beta(\mu - \varepsilon_k)}\right) \quad ; \quad \eta = \begin{cases} +1, & \text{bosons} \\ -1, & \text{fermions} \end{cases}$$

$$= \sum_k \eta k_B T \ln\left(1 - \eta e^{-\beta(\varepsilon_k - \mu)}\right) \quad \text{---- (1)}$$

now

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = -\sum_k \left[\eta k_B \ln\left(1 - \eta e^{-\beta(\varepsilon_k-\mu)}\right) + \frac{\eta k_B T \left(\frac{\varepsilon_k - \mu}{k_B}\right)\left(\frac{-1}{T^2}\right) e^{-\beta(\varepsilon_k - \mu)}}{1 - \eta e^{-\beta(\varepsilon_k-\mu)}}\right]$$

$$= \sum_k \left[\eta k_B \ln\left(1 - \eta e^{-\beta(\varepsilon_k-\mu)}\right) - \frac{k_B \beta(\varepsilon_k - \mu)\, e^{-\beta(\varepsilon_k-\mu)}}{1 - \eta e^{-\beta(\varepsilon_k-\mu)}}\right]$$

let $x = e^{-\beta(\varepsilon_k - \mu)}$

$\ln x = -\beta(\varepsilon_k - \mu)$

$$\Rightarrow S = -\sum_k \left[\eta k_B \ln(1 - \eta x) + \frac{(k_B \ln x)\, x}{1 - \eta x}\right]$$

$$= -k_B \sum_k \frac{\eta(1 - \eta x)\ln(1 - \eta x) + x \ln x}{(1 - \eta x)} \quad ; \quad \eta^2 = +1$$

$$= -k_B \sum_k \frac{\eta \ln(1 - \eta x) - x \ln(1 - \eta x) + x \ln x}{(1 - \eta x)}$$

$$= k_B \sum_k \frac{-\eta \ln(1 - \eta x) + x \ln(1 - \eta x) - x \ln x}{(1 - \eta x)}$$

$$S = k_B \sum_k \frac{\gamma \ln\left(\frac{1}{1-\gamma x}\right) + x \ln\left(\frac{1-\gamma x}{x}\right)}{(1-\gamma x)}$$

$$= k_B \left[\sum_k \frac{\gamma}{1-\gamma x} \ln\left(\frac{1}{1-\gamma x}\right) + \sum_k \left(\frac{x}{1-\gamma x}\right) \ln\left(\frac{1-\gamma x}{x}\right) \right] \quad \text{--- (2)}$$

Now $\bar{n}_\varepsilon = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - \gamma} = \frac{1}{\frac{1}{x} - \gamma}$; where $x = e^{-\beta(\varepsilon_k - \mu)}$

$$= \frac{x}{1-\gamma x} \Rightarrow \boxed{\frac{1-\gamma x}{x} = \frac{1}{\bar{n}_\varepsilon}} \Rightarrow \frac{1}{x} - \gamma = \frac{1}{\bar{n}_\varepsilon}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{\bar{n}_\varepsilon} + \gamma = \frac{1+\gamma \bar{n}_\varepsilon}{\bar{n}_\varepsilon} \Rightarrow x = \frac{\bar{n}_\varepsilon}{1+\gamma \bar{n}_\varepsilon}$$ ⇒ relabel

$$\bar{n}_\varepsilon \equiv n_\varepsilon \Rightarrow \boxed{x = \frac{n_\varepsilon}{1+\gamma n_\varepsilon}}$$

now substitute in (2)

$$S = k_B \left[\sum_k \frac{\gamma}{1 - \frac{\gamma n_\varepsilon}{1+\gamma n_\varepsilon}} \ln\left(\frac{1}{1 - \frac{\gamma n_\varepsilon}{1+\gamma n_\varepsilon}}\right) + \sum_k n_\varepsilon \ln\left(\frac{1}{n_\varepsilon}\right) \right]$$

$$= k_B \sum_k \left(\gamma(1+\gamma n_\varepsilon) \ln(1+\gamma n_\varepsilon) - n_\varepsilon \ln n_\varepsilon\right)$$

- for bosons, $\gamma = +1$

$$S = k_B \sum_k \left[(1+n_\varepsilon)\ln(1+n_\varepsilon) - n_\varepsilon \ln n_\varepsilon\right] \quad \text{--- (3)}$$

- for fermions, $\gamma = -1$

$$S = k_B \sum_k \left[-(1-n_\varepsilon)\ln(1-n_\varepsilon) - n_\varepsilon \ln n_\varepsilon\right] \quad \text{--- (4)}$$

Alternative method

- for fermions, $P_\varepsilon(n) = \begin{cases} \bar{n}_\varepsilon & ; \text{for } n=1 \\ 1-\bar{n}_\varepsilon & ; \text{for } n=0 \end{cases}$

See equations 6.3.10 and 6.3.11 of Pathria

$$\Rightarrow \quad S = -k_B \sum_\varepsilon \sum_n P_\varepsilon(n) \ln P_\varepsilon(n)$$

Note: $\bar{n}_\varepsilon \equiv n_\varepsilon$

$$= -k_B \sum_\varepsilon (1-\bar{n}_\varepsilon)\ln(1-\bar{n}_\varepsilon) + \bar{n}_\varepsilon \ln \bar{n}_\varepsilon \quad ; \text{ same as (4)}$$

- for bosons, $\bar{n}_\varepsilon = \dfrac{1}{\frac{1}{x}-1} \Rightarrow x = \dfrac{\bar{n}_\varepsilon}{1+\bar{n}_\varepsilon}$; $x = e^{-\beta(\varepsilon_k-\mu)}$

again let $\bar{n}_\varepsilon \to n_\varepsilon$ (relabel) $\Rightarrow \boxed{x = \dfrac{n_\varepsilon}{1+n_\varepsilon}}$

$$P_\varepsilon(n) = \frac{x^n}{\sum_n x^n} = \frac{x^n}{\frac{1}{1-x}} = x^n(1-x) \quad ; \text{ where } \sum x^n = \frac{1}{1-x}$$

$$= \left(\frac{n_\varepsilon}{1+n_\varepsilon}\right)^n \left(1-\frac{n_\varepsilon}{1+n_\varepsilon}\right) = \left(\frac{n_\varepsilon}{1+n_\varepsilon}\right)^n \left(\frac{1}{1+n_\varepsilon}\right)$$

$$= \frac{n_\varepsilon^n}{(1+n_\varepsilon)^{n+1}}$$

now $S = -k_B \sum_n P_\varepsilon(n) \ln P_\varepsilon(n)$

$$= -k_B \sum_n P_\varepsilon(n) \ln \frac{n_\varepsilon^n}{(1+n_\varepsilon)^{n+1}}$$

$$= -k_B \left[\sum_n P_\varepsilon(n)\, n \ln n_\varepsilon - P_\varepsilon(n)(n+1)\ln(1+n_\varepsilon)\right]$$

$$= -k_B \ln n_\varepsilon \underbrace{\sum_n n P_\varepsilon(n)}_{\langle n_\varepsilon\rangle = \bar{n}_\varepsilon \equiv n_\varepsilon} + k_B \ln(1+n_\varepsilon) \underbrace{\sum_n P_\varepsilon(n)\,(n_\varepsilon+1)}_{\langle \bar{n}_\varepsilon+1\rangle \equiv n_\varepsilon+1}$$

$$S = -k_B \bar{n}_\varepsilon \ln \bar{n}_\varepsilon + k_B(\bar{n}_\varepsilon+1)\ln(\bar{n}_\varepsilon+1)$$

same as (3)

② $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$ geometric series —— (1)

a) taking the first derivative of (1), gives

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$ —— (2) ; here n can not start from zero

$$\sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2} \Rightarrow \sum_0 n x^n + \underbrace{\sum_0 x^n}_{\frac{1}{1-x}} = \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} n x^n + \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} n x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{1}{1-x}\left[\frac{1}{1-x} - 1\right] = \frac{x}{(1-x)^2}$$

b) taking the second derivative of (1), gives

$$\sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}$$; n starts from 2

$$\sum_{n=0}^{\infty} (n+2)(n+1) x^n = \frac{2}{(1-x)^3} \Rightarrow \sum_{n=0}^{\infty} (n^2+3n+2) x^n = \frac{2}{(1-x)^3}$$

$$\Rightarrow \sum_{n=0}^{\infty} n^2 x^n + 3\sum_{n=0} n x^n + 2\sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} n^2 x^n + \frac{3x}{(1-x)^2} + \frac{2}{1-x} = \frac{2}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{2}{(1-x)^3} - \frac{3x}{(1-x)^2} - \frac{2}{(1-x)}$$

$$= \frac{1}{(1-x)^3}\left[2 - 3x(1-x) - 2(1-x)^2\right]$$

$$= \frac{1}{(1-x)^3}\left[x + x^2\right] = \frac{x(x+1)}{(1-x)^3}$$ Q.E.D

③ Pathria 6.2

$$P_\varepsilon(n) = \begin{cases} \dfrac{\bar{n}_\varepsilon^{\,n}}{(\bar{n}_\varepsilon+1)^{n+1}} & ; \text{ B.E} \\ \left.\begin{matrix} \bar{n}_\varepsilon \ ; \ n=1 \\ 1-\bar{n}_\varepsilon \ ; \ n=0 \end{matrix}\right\} & ; \text{ F.D} \\ \dfrac{\bar{n}_\varepsilon^{\,n}}{n!} e^{-\bar{n}_\varepsilon} & ; \text{ M.B} \end{cases}$$

– $\boxed{\text{B.E}}$ $P_\varepsilon(n) = \dfrac{\bar{n}_\varepsilon^{\,n}}{(\bar{n}_\varepsilon+1)^{n+1}}$; let $r = \dfrac{\bar{n}_\varepsilon}{\bar{n}_\varepsilon+1}$

$$\Rightarrow \frac{1}{r} = \frac{\bar{n}_\varepsilon+1}{\bar{n}_\varepsilon} = 1+\frac{1}{\bar{n}_\varepsilon}$$

$$\Rightarrow \frac{1}{r} - 1 = \frac{1}{\bar{n}_\varepsilon}$$

$$\Rightarrow \frac{1-r}{r} = \frac{1}{\bar{n}_\varepsilon}$$

$$\Rightarrow \boxed{\bar{n}_\varepsilon = \frac{r}{1-r}} \quad \text{---(1)}$$

and $\bar{n}_\varepsilon + 1 = \dfrac{1}{1-r}$

$$\Rightarrow P_\varepsilon(n) = \frac{r^n}{(1-r)^2} \Big/ \frac{1}{(1-r)^{n+1}}$$

$$= (1-r)\, r^n$$

$$\langle n_\varepsilon \rangle = \sum_n n_\varepsilon P_\varepsilon(n) \ ; \text{ here } n_\varepsilon = n$$

where n_ε: # of bosons in state with energy ε

$P_\varepsilon(n)$: probability of state ε to have n bosons

$$\langle n_\varepsilon \rangle = \sum_n n_\varepsilon P_\varepsilon(n) \ ; \quad n_\varepsilon = n$$

$$= \sum_{n=0}^{\infty} n\, P_\varepsilon(n) = (1-r) \underbrace{\sum_{n=0}^{\infty} n r^n}_{\frac{r}{(1-r)^2}} = (1-r)\frac{r}{(1-r)^2} = \frac{r}{1-r} \quad \text{--(2)}$$

and $\langle n_\varepsilon^2 \rangle = \sum_n n_\varepsilon^2 P_\varepsilon(n)$; $n_\varepsilon = n$

$= \sum_n n^2 P_\varepsilon(n)$; $P_\varepsilon(n) = (1-r)\, r^n$

$$= (1-r) \underbrace{\sum_{n=0}^{\infty} n^2 r^n}_{\frac{r(r+1)}{(1-r)^3}} = (1-r)\frac{r(r+1)}{(1-r)^3} = \frac{r(r+1)}{(1-r)^2}$$

$$\therefore \langle n_\varepsilon^2 \rangle - \langle n_\varepsilon \rangle^2 = \frac{r(r+1)}{(1-r)^2} - \frac{r^2}{(1-r)^2} = \frac{r}{(1-r)^2}$$

$$= \frac{\bar{n}_\varepsilon}{\bar{n}_\varepsilon + 1}(\bar{n}_\varepsilon + 1)^2 = \bar{n}_\varepsilon(\bar{n}_\varepsilon + 1)$$

$$= \bar{n}_\varepsilon^2 + \bar{n}_\varepsilon$$

$$= \langle n_\varepsilon \rangle^2 + \langle n_\varepsilon \rangle \quad \text{--- (3)}$$

\- $\boxed{\text{F.D}}$ $P_\varepsilon(n) = \bar{n}_\varepsilon$; $n=1$

$= 1 - \bar{n}_\varepsilon$; $n=0$

$\Rightarrow \langle n_\varepsilon \rangle = \sum_{n=0}^{1} n_\varepsilon P_\varepsilon(n)$; $n_\varepsilon = n$

$= \sum_{n=0}^{1} n P_\varepsilon(n) = 0 + P_\varepsilon(n=1) = \bar{n}_\varepsilon$

and $\langle n_\varepsilon^2 \rangle = \sum_{n=0}^{1} n^2 P_\varepsilon(n) = 0 + P_\varepsilon(n=1) = \bar{n}_\varepsilon$

$$\therefore \boxed{\langle n_\varepsilon^2 \rangle - \langle n_\varepsilon \rangle^2 = \bar{n}_\varepsilon - \bar{n}_\varepsilon^2 = \langle n_\varepsilon \rangle - \langle n_\varepsilon \rangle^2} \quad \text{--- (4)}$$

$-\ \boxed{M.B}\qquad P_\varepsilon(n) = \dfrac{\bar{n}_\varepsilon^{\,n}}{n!}\, e^{-\bar{n}_\varepsilon}$

$$\langle n_\varepsilon(n_\varepsilon - 1)\rangle = \sum_n n_\varepsilon(n_\varepsilon-1)\, P_\varepsilon(n) \quad ; \quad n_\varepsilon = n$$

$$= \sum_n n(n-1)\, P_\varepsilon(n) = \sum_n n(n-1)\, \frac{\bar{n}_\varepsilon^{\,n}}{n!}\, e^{-\bar{n}_\varepsilon}$$

$$= e^{-\bar{n}_\varepsilon} \sum \cancel{n(n-1)}\, \frac{\bar{n}_\varepsilon^{\,n}}{\cancel{n(n-1)}(n-2)!} = e^{-\bar{n}_\varepsilon} \sum_{n=3}^{\infty} \frac{\bar{n}_\varepsilon^{\,n}}{(n-2)!}$$

$$= e^{-\bar{n}_\varepsilon} \sum_{n=0}^{\infty} \frac{\bar{n}_\varepsilon^{\,n+3}}{(n+1)!} = e^{-\bar{n}_\varepsilon}\, \bar{n}_\varepsilon^{\,2}\, \underbrace{\sum_{n=0}^{\infty} \frac{\bar{n}_\varepsilon^{\,n+1}}{(n+1)!}}_{e^{+\bar{n}_\varepsilon}}$$

$$= \bar{n}_\varepsilon^{\,2} = \langle n_\varepsilon \rangle^2$$

$$\therefore\ \langle n_\varepsilon(n_\varepsilon - 1)\rangle = \langle n_\varepsilon^2 - n_\varepsilon\rangle = \langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle$$

$$\Downarrow$$

$$\langle n_\varepsilon\rangle^2 = \langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle$$

$$\Rightarrow \boxed{\langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 = \langle n_\varepsilon\rangle} \quad \text{———} \quad (5)$$

Now in general $\langle n_\varepsilon\rangle = \dfrac{1}{e^{(\varepsilon-\mu)/k_BT} - \mathcal{J}}$; $\mathcal{J} = \begin{cases} +1, & \text{bosons} \\ -1, & \text{Fermions} \\ 0, & \text{M.B} \end{cases}$

$\Rightarrow \langle n_\varepsilon\rangle^{-1} = e^{(\varepsilon-\mu)/k_BT} - \mathcal{J} \quad \Rightarrow$ diff both sides w.r.t μ

$$\Rightarrow -\langle n_\varepsilon\rangle^{-2} \left(\frac{\partial\langle n_\varepsilon\rangle}{\partial\mu}\right)_T = -\frac{1}{k_BT}\, e^{(\varepsilon-\mu)/k_BT} = -\frac{1}{k_BT}\left[\langle n_\varepsilon\rangle^{-1} + \mathcal{J}\right]$$

$$\Rightarrow k_BT\left(\frac{\partial\langle n_\varepsilon\rangle}{\partial\mu}\right)_T = \langle n_\varepsilon\rangle + \mathcal{J}\langle n_\varepsilon\rangle^2 \quad \text{———} \quad (6)$$

Now using equation 6.3.9 Pathria Page 151

$$\frac{\langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2}{\langle n_\varepsilon\rangle^2} = \frac{1}{\langle n_\varepsilon\rangle} + \gamma \quad ; \quad$$ pathria uses a instead of γ

$$\Rightarrow \langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 = \langle n_\varepsilon\rangle + \gamma \langle n_\varepsilon\rangle^2$$

we substitute this back in equation (6), we get

$$k_B T \left(\frac{\partial \langle n_\varepsilon\rangle}{\partial \mu}\right)_T = \langle n_\varepsilon\rangle + \gamma \langle n_\varepsilon\rangle^2$$

$$= \langle n_\varepsilon^2\rangle - \langle n_\varepsilon\rangle^2 \qquad \text{(Q. 6. 1)}$$

(4) Quantum gas in d dimensions with $\varepsilon = c|\vec{k}|^a$;

a) $\Omega = \sum_k \Omega_k = \sum_k \eta k_B T \ln(1-\eta e^{\beta(\mu-\varepsilon_k)})$; $\eta = \begin{cases} +1, & \text{Bos} \\ -1, & \text{Fer} \end{cases}$

$$= \frac{\eta k_B T}{(2\pi)^d} \int d^d q \, d^d k \ln(1-\eta e^{\beta(\mu-ck^a)})$$

$$= \eta k_B T \frac{L^d}{(2\pi)^d} \int C_d k^{d-1} dk \ln(1-\eta e^{\beta(\mu-ck^a)})$$

Note that in 3d $d^3k = 4\pi k^2 dk$ $\quad 0<k<\infty$

in 2d $d^2k = k\,dk^1\,d\theta$ $\quad 0\leq\theta\leq 2\pi$

in 1d $dk = k^0 dk$

so in d dimensions $dk^d = C_d k^{d-1} dk$

Now $\Omega = \eta k_B T \frac{L^d}{(2\pi)^d} C_d \int_0^\infty k^{d-1} dk \ln(1-\eta e^{\beta(\mu-ck^a)})$

integrate by parts $u = \ln(1-\eta e^{\beta(\mu-ck^a)})$, $dv = k^{d-1} dk$

$$du = \frac{a\eta\beta c k^{a-1} e^{\beta(\mu-ck^a)}}{1-\eta e^{\beta(\mu-ck^a)}} \qquad v = \frac{k^d}{d}$$

$$\Rightarrow \Omega = \eta k_B T \frac{L^d}{(2\pi)^d} C_d \left[\frac{1}{d} \cancel{k^d \ln(1-\eta e^{\beta(\mu-ck^a)})}^{\,0} \Big|_0^\infty \right.$$

$$\left. - \int \frac{a\eta\beta c k^{d+a-1} dk \, e^{\beta(\mu-ck^a)}}{d \, (1-\eta e^{\beta(\mu-ck^a)})} \right.$$

$$\boxed{\Omega = \frac{-L^d}{(2\pi)^d} C_d \frac{c}{d} a \int_0^\infty \frac{dk\, k^{d+a-1} e^{\beta(\mu-ck^a)}}{1-\eta e^{\beta(\mu-ck^a)}}} \quad \text{--- (1)}$$

b) $U = \sum_k \varepsilon_k \bar{n}_k$; $\bar{n}_k = \dfrac{1}{e^{\beta(\varepsilon_k - \mu)} - \gamma}$

$$U = \frac{L^d}{(2\pi)^d} \int_0^\infty \frac{C_d\, k^{d-1}\, dk \;\; c k^a}{e^{\beta(\varepsilon_k - \mu)} - \gamma} = \frac{L^d}{(2\pi)^d} C_d\, c \int_0^\infty \frac{k^{a+d-1}\, dk}{e^{\beta(\varepsilon_k - \mu)} - \gamma}$$

$$= \frac{L^d}{(2\pi)^d} C_d\, c \underbrace{\int_0^\infty \frac{k^{a+d-1}\, dk\, e^{\beta(\mu - ck^a)}}{1 - \gamma e^{\beta(\mu - ck^a)}}}_{\text{from (1), } = -\Omega \frac{(2\pi)^d}{L^d} \frac{d}{C_d c a}}$$

$$= \frac{L^d}{(2\pi)^d} C_d\, c \left(-\Omega \frac{(2\pi)^d}{L^d} \frac{d}{C_d c a} \right) = -\frac{\Omega d}{a} = \frac{d}{a}(-\Omega)$$

$$U = +\frac{d}{a}(PV) \implies \boxed{PV = \frac{a}{d} U}$$ equation of state

check for 3d with $\varepsilon \sim k^2$ $\Rightarrow PV = \frac{2}{3} U$ as expected for classical non interacting gas

c) $$N = \sum_k n_k = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - \gamma} = \frac{L^d}{(2\pi)^d} \int_0^\infty \frac{C_d\, k^{d-1}\, dk}{e^{\beta(\varepsilon_k - \mu)} - \gamma}$$

$$= \frac{L^d}{(2\pi)^d} C_d \int_0^\infty \frac{k^{d-1}\, dk}{e^{\beta(ck^a - \mu)} - \gamma} = \frac{L^d}{(2\pi)^d} C_d \int_0^\infty \frac{k^{d-1}\, dk}{e^{-\beta\mu} e^{\beta c k^a} - \gamma}$$

$$N = \frac{L^d}{(2\pi)^d} C_d \int_0^\infty \frac{k^{d-1}\, dk}{z^{-1} e^{\beta c k^a} - \gamma}$$; when $z = e^{\beta\mu}$ --- (2)

In the high T limit

$\beta\mu \ll 1 \Rightarrow e^{\beta\mu} \ll e$ or $z^{-1} \gg 1$; when $z = e^{\beta\mu}$

$\Rightarrow$ in eqⁿ (2), $z^{-1} e^{\beta c k^a} \gg \eta = \pm 1$

$$\Rightarrow N = \frac{L^d}{(2\pi)^d} C_d\, z \int_0^\infty dk\, k^{d-1} e^{-\beta c k^a} \quad --- \quad (3)$$

for 3D classical ideal gas $\varepsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \equiv c k^a$

$\Rightarrow c = \frac{\hbar^2}{2m}$ and $a = 2$

$$\Rightarrow N = \frac{V}{(2\pi)^3} C_d\, z \int_0^\infty dk\, k^2 e^{-\beta c k^2}$$

, let $\beta c k^2 = x^2$

$2\beta c k\, dk = 2x\, dx$

$k^2 dk = \frac{x^2 dx}{(\beta c)^{3/2}}$

$$= \frac{V}{(2\pi)^3} C_d \underbrace{e^{\beta\mu}}_{z} \frac{1}{(\beta c)^{3/2}} \underbrace{\int_0^\infty dx\, x^2 e^{-x^2}}_{\sqrt{\pi}/4}$$

$$= \frac{V}{8\pi^3} C_d\, e^{\beta\mu} \frac{1}{(\beta c)^{3/2}} \frac{\sqrt{\pi}}{4}$$

; $C_d = 4\pi$, $d^3k = \underbrace{4\pi}_{C_d} k^2 dk$

$$= \frac{V}{8\pi^3} 4\pi e^{\beta\mu} \frac{1}{\left(\frac{\hbar^2 c}{k_B T\, 2m}\right)^{3/2}} \frac{\sqrt{\pi}}{4} = \frac{V}{8\pi^3} \frac{8\pi^3}{h^3} e^{\beta\mu} (2\pi m k_B T)^{3/2}$$

$$= \frac{V}{h^3/(2\pi m k_B T)^{3/2}} e^{\beta\mu} = \frac{V}{\lambda^3} e^{\beta\mu}$$

as expected from equation 6.10 in our lecture notes