Normal Zeeman Effect ($S=0$)

the classical hamiltonian of the H atom is $\hat{H}_0$

$$\hat{H}_0 = \frac{\hat{P}^2}{2m} + V(r) \quad ; \quad V(r) = \frac{-e^2}{r}$$ central Potential

with $$E_n^{(0)} = -\frac{me^4}{2\hbar^2}\frac{1}{n^2} = -\frac{13.6}{n^2}\ \text{eV}$$

the unperturbed eigenstates are $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$
$= |nlm\rangle$

now when the H atom is placed in an external uniform magnetic field, its energy levels get shifted. This energy shift is called the Zeeman effect. in this problem we ignore the spin first ($s=0$). this is called norman Zeeman effect. when the spin is taken into account (later), we get what is called the anomalous Zeeman effect.

let $\vec{B} = B\hat{k}$, $H' = -\vec{\mu}_l \cdot \vec{B}$; where $\vec{\mu}_l = \frac{-e}{2mc}\vec{L}$

in general $\vec{\mu}_L = \frac{q}{2mc}\vec{L}$ orbital magnetic moment

so if q is positive $\Rightarrow$ $\vec{\mu}_L$ and $\vec{L}$ are parallel

if q is negative $\Rightarrow$ " " " antiparallel

$$\Rightarrow \hat{H}' = \frac{e}{2mc}\hat{\vec{L}}\cdot\vec{B} = \frac{eB}{2mC}\hat{L}_z$$; H': perturbation

$H = H_0 + H'$

$$\Rightarrow E_n^{(1)} = \langle nlm|H'|nlm\rangle = \frac{eB}{2mc}\langle nlm|L_z|nlm\rangle$$
$$= \frac{eB}{2mc} m_l\hbar \quad ; \quad -l \le m_l \le l$$
$(2l+1)$ states

notice that (see back) H_0 commute with L_z and hence with H' $\Rightarrow$ the operators $\hat{H}'$, L_z, H_0 commute $\Rightarrow$ can be represented by same eigenstates $|nlm\rangle$

∴ $E_n^{(1)} = \frac{e\hbar}{2mc} m_\ell B = \mu_B m_\ell B$; where μ_B : Bhor magneton

$$\mu_B = \frac{e\hbar}{2mc} = 9.27\times10^{-24}\ JT^{-1}$$
$$= 5.79\times10^{-5}\ eV.T^{-1}$$

so the energy to first order is

$$E_{n\ell m} = E_n^{(0)} + E_n^{(1)} = E_n^{(0)} + \mu_B m_\ell B$$

so when the H atom is placed in a uniform magnetic field, and if we ignore the spin, each level with angular momentum ℓ will split into $(2\ell+1)$ sublevels (equally spaced)

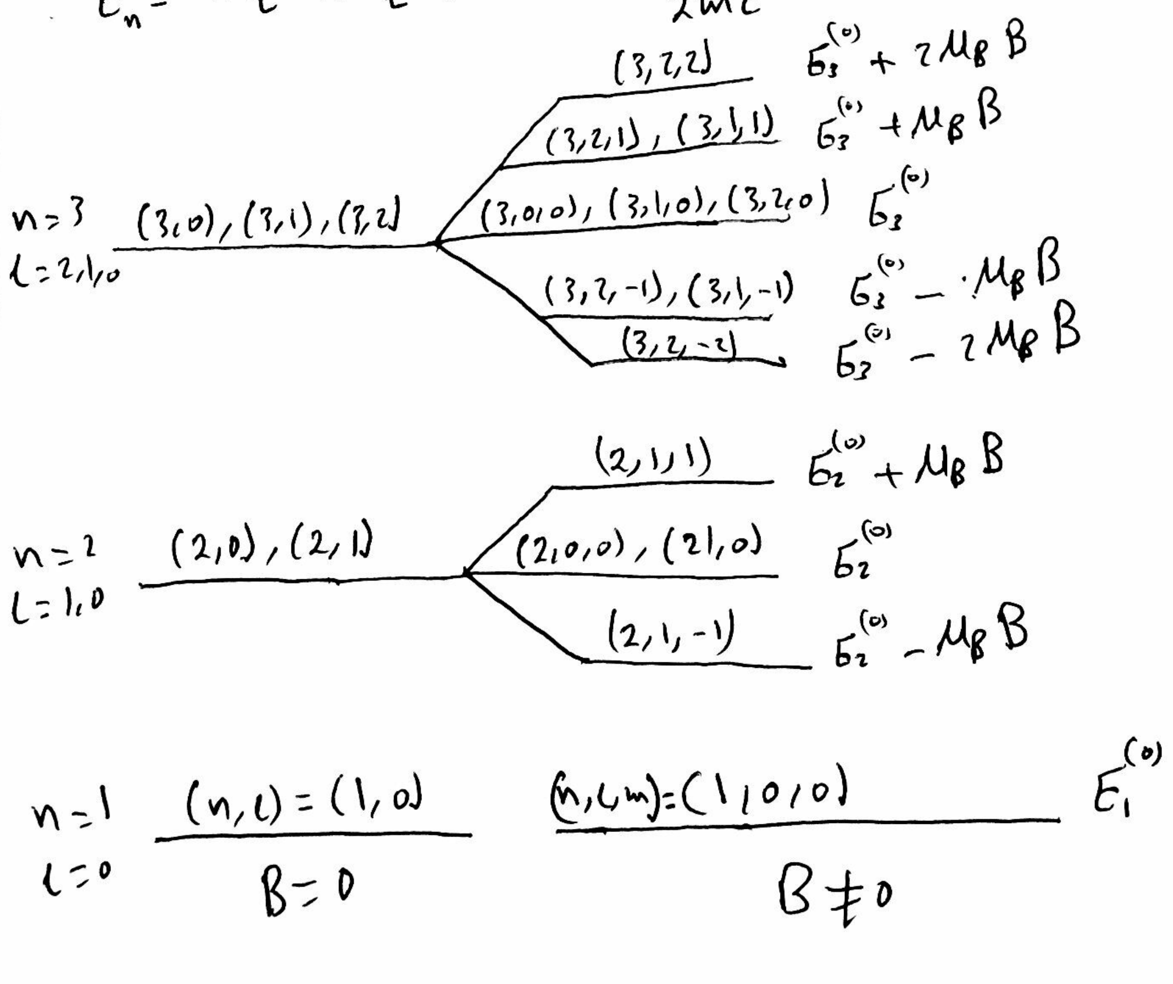
i.e. $\Delta E = E_{n\ell,m+1} - E_{n\ell m}$
$$= \left(E_n^{(0)} + \mu_B (m+1) B\right) - \left(E_n^{(0)} + \mu_B m B\right) = \mu_B B$$

$\Rightarrow \Delta E \propto B$ independent of ℓ

This equidistant splitting is know as normal (Zeeman effect)

Notice that $E_n^{(1)}$ can be written as

$E_n^{(1)} = m_\ell \hbar \omega_L$; $\omega_L = \frac{eB}{2mc}$ Larmor frequency.



the normal Zeeman effect has removed the degeneracy only partially. we see that each energy level splits into an odd number of $(2l+1)$ equally spaced levels. This however disagree with experimental observation. for instance, each energy level in the H atom splits into an even # of levels, indicating that the angular momentum l in $(2l+1)$ is Not integer but half integer. This disagreement is due to the ignorance of the spin of the electron. Actully, the angular momentum of the electron is not pure orbital but it has a spin component as well. This leads to the splitting of each level into an even # of $(2j+1)$ unequally spaced energy levels. This effect is known as Anomalous Zeeman effect, which was observed and verified experimentally.

- The Anomalous Zeeman Effect:

Now we are going to take the electron's spin into account in calculating the energy corrections.

$$H' = -\vec{\mu}_L \cdot \vec{B} - \vec{\mu}_s \cdot \vec{B} = \frac{e}{2mc} \vec{L} \cdot \vec{B} + \frac{e}{mc} \vec{S} \cdot \vec{B} = \frac{e}{2mc} (\vec{L} + 2\vec{S}) \cdot \vec{B}$$

$$; \quad \vec{\mu}_L = \frac{-e}{2mc} \vec{L}$$

let $\vec{B} = B\hat{k} \Rightarrow H'_z = \dfrac{eB}{2mc} (\hat{L}_z + 2\hat{S}_z)$; $\vec{\mu}_s = \dfrac{-e}{mc} \vec{S}$

where $\hat{H}_0 = \dfrac{p^2}{2m} - \dfrac{e^2}{r}$

the total Hamiltonian is

$$H = H_0 + \underbrace{H'_{SO}}_{\text{spin-orbit}} + \underbrace{H'_z}_{\text{Zeeman correction}} + \underbrace{H_R}_{\text{Relativistic corrections}}$$

$$\Rightarrow H = \underbrace{H_0}_{\text{unperturbed}} + \underbrace{H'_{FS}}_{\substack{\text{Fine structure} \\ \text{(Relativistic + spin orbit)}}} + \underbrace{H'_Z}_{\text{Zeeman correction}} \quad \text{---} (10)$$

now let us study this Hamiltonian in two limits.

a) The strong Field Zeeman effect:

We know that $H'_Z = \frac{eB}{2mc}(L_z + 2S_z)$;

and $H'_{FS} = H'_{SO} + H'_R = \frac{e^2}{2m^2c^2r^3}\vec{L}\cdot\vec{S} + \cancel{H_R}$ (very small)

$\Rightarrow \frac{H'_Z}{H'_{FS}} = (\text{constant}) B$

so for very high fields $H'_Z >> H'_{FS}$ see back

so H is reduced to $\hat{H} = \hat{H}_0 + \hat{H}'_Z = \hat{H}_0 + \frac{eB}{2mc}(\hat{L}_z + 2\hat{S}_z)$

here $[H, H_0] = 0$ because $[H_0, L_z] = [H_0, S_z] = 0$

then H may be represented by a common set of states $|nlm_lm_s\rangle$; in these bases L^2, S^2, L_z, S_z form C.S.C.O

$\therefore E_n^{(1)} = \langle nlm_lm_s | H'_Z | nlm_lm_s \rangle = \frac{eB}{2mc}(\hbar m_l + 2\hbar m_s)$

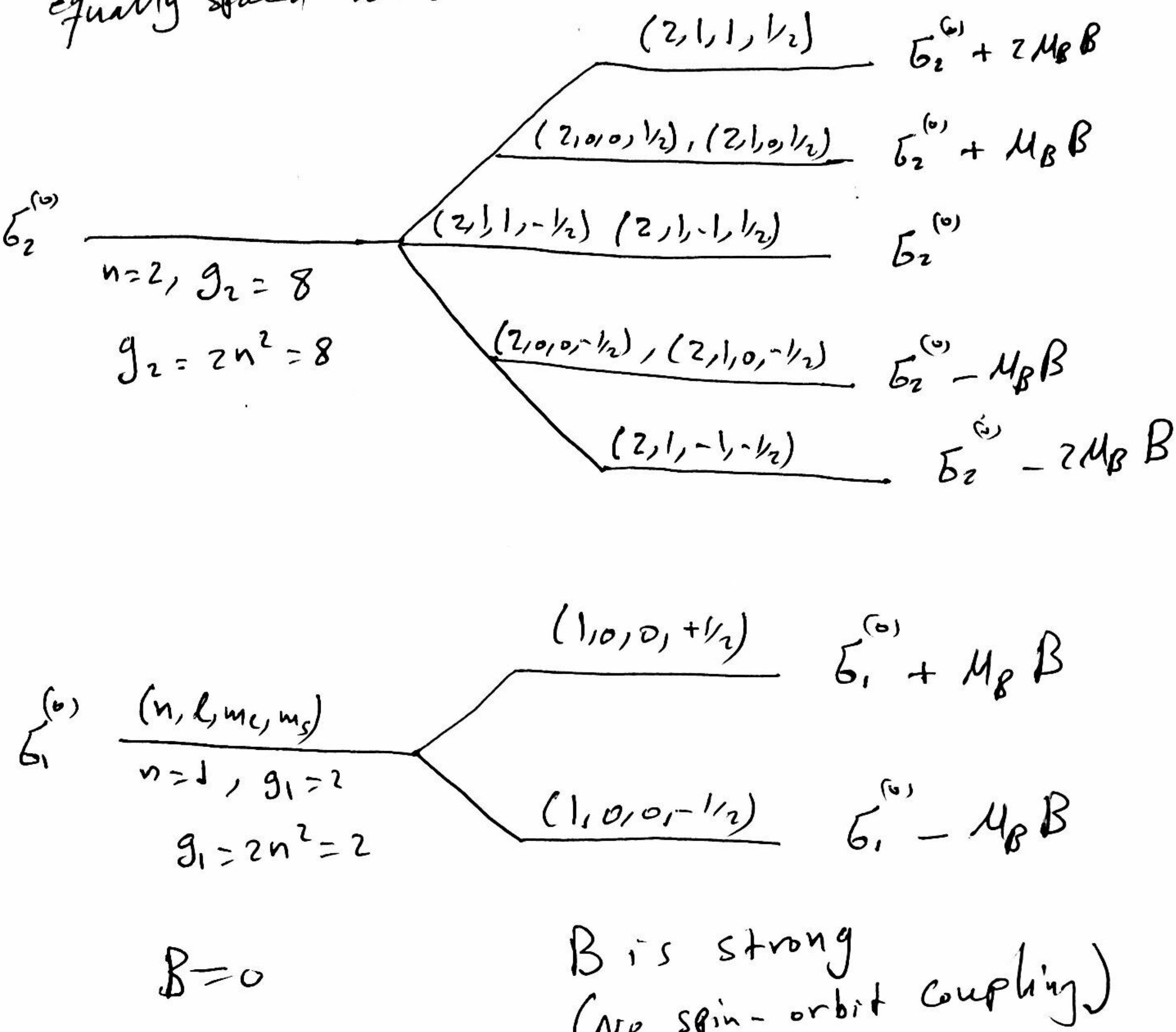
$\Rightarrow E^{(1)}_{nlm_lm_s} = E_n^{(0)} + \frac{eB\hbar}{2mc}(m_l + 2m_s)$; $E_n^{(0)} = \frac{-e^2}{2a_0n^2}$

$$= \frac{-e^2}{2a_0n^2} + \frac{eB\hbar}{2mc}(m_l + 2m_s)$$

- the energy levels are thus shifted by an amount of

$\Delta E = \frac{eB\hbar}{2mc}(m_l + 2m_s) = \mu_B B(m_l + 2m_s)$; $\mu_B = \frac{e\hbar}{2mc}$

equally spaced levels



b) The weak field Zeeman effect:

if B is weak, then we need to consider all terms in the total Hamiltonian (equation 10). now the good eigenstates to use are the $|nljm_j\rangle$ states.

now writing $L_z + 2S_z$ as $(L_z + S_z) + S_z = J_z + S_z$

$$\Rightarrow \hat{H} = \hat{H}_0 + \hat{H}'_{FS} + H'_z \quad ; \text{ where}$$

$$= \hat{H}_0 + \hat{H}'_{FS} + \frac{eB}{2mc}(\hat{J}_z + \hat{S}_z)$$

; $\langle nljm_j | H_{FS} | nljm_j\rangle$ was found in the last section of FS corrections see back

in the first order approximation

$$\langle nljm_j | H'_z | nljm_j\rangle = \frac{eB}{2mc}\langle nljm_j | J_z + S_z | nljm_j\rangle$$

now $\langle n\ell j m_j | J_z | n\ell j m_j \rangle = \hbar m_j$

and I am left to find $\langle n\ell j m_j | S_z | n\ell j m_j \rangle$?

now $\vec{J} = \vec{L} + \vec{S} \Rightarrow J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S} \Rightarrow \vec{L}\cdot\vec{S} = \dfrac{J^2 - L^2 - S^2}{2}$

also $\vec{J}\cdot\vec{S} = (\vec{L}+\vec{S})\cdot\vec{S} = \vec{L}\cdot\vec{S} + S^2$

$$= \frac{J^2 - L^2 - S^2}{2} + S^2 = \frac{J^2 - L^2}{2} + \frac{S^2}{2}$$

$$= \frac{J^2 - L^2 + S^2}{2}$$

notice that $|n\ell j m_j\rangle$ are joint eigenstates of J^2, J_z, L^2, S^2

now from Wigner–Eckart theorem (see chapter 7 eq^n 7.337)

$$\langle n\ell j m_j | S_z | n\ell j m_j \rangle = \frac{\langle n\ell j m_j | \vec{J}\cdot\vec{S} | n\ell j m_j \rangle}{\hbar^2 j(j+1)} \langle n\ell j m_j | J_z | n\ell j m_j \rangle$$

$$= \frac{\cancel{\hbar^2}[j(j+1) - \ell(\ell+1) + s(s+1)]}{\cancel{\hbar^2} j(j+1)} \hbar m_j$$

$=$

$$\Rightarrow \langle n\ell j m_j | H_z' | n\ell j m_j \rangle = \frac{eB\hbar}{2mc} \underbrace{\left[1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right]}_{g_j} m_j$$

$$= \frac{eB\hbar}{2mC} g_j m_j$$

when $\ell = 0 \Rightarrow j = s \Rightarrow g_s = 2$

when $s = 0 \Rightarrow j = \ell \Rightarrow g_\ell = 1$

where g_j is called Landé factor

$$= \mu_B g_j m_j B \; ; \; \mu_B = \frac{e\hbar}{2mC} \text{ (in CGS)} = \frac{e\hbar}{2m} \text{ (in SI)}$$

for $s = 1/2$ and $j = \ell \pm 1/2$

$$g_{j=\ell\pm 1/2} = 1 \pm \frac{1}{2\ell+1} = \begin{cases} \dfrac{2\ell+2}{2\ell+1} & ; \; j = \ell + 1/2 \\[2ex] \dfrac{2\ell}{2\ell+1} & ; \; j = \ell - 1/2 \end{cases}$$

so each level j is split into an even # of $(2j+1)$ sublevels corresponding to $m_j = -j, -j+1, \ldots, j-1, j$

Notice that the spacing between the sublevels corresponding to $j = l - 1/2$ is $\Delta E_1 = \mu_B B \frac{2l}{2l+1}$ see back, and for $j = l + 1/2$ ΔE_2

$\Delta E_2 = \mu_B B \frac{2l+2}{2l+1}$. We see that $\Delta E_1 \neq \Delta E_2$ which is for the same value of l; in contrast with the normal Zeeman effect, when the spacing between sublevels of the same l is fixed. Therefore, this effect is called Anolamous Zeeman effect.