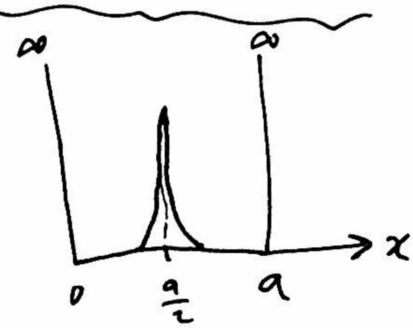
Graduate QM

HW # 4 - Solution

Dr. Gassem Alzoubi

① $H_0 = \frac{p^2}{2m}$; $H_0 \psi_n^{(0)}(x) = E_n^{(0)} \psi_n^{(0)}(x)$



$E_n^{(0)} = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$; $\psi_n^{(0)}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$

$H' = \alpha\, \delta(x - \frac{a}{2})$

a) $E_n^{(1)} = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \frac{2\alpha}{a} \int_0^a \sin^2 \frac{n\pi}{a}x \;\delta(x-\frac{a}{2})\, dx$

$= \frac{2\alpha}{a} \sin^2\left(\frac{n\pi}{a}\frac{a}{2}\right) = \frac{2\alpha}{a} \sin^2\left(n\frac{\pi}{2}\right)$

$= \begin{cases} 0 \;; & \text{if } n \text{ is even} \\ \frac{2\alpha}{a} \;; & \text{if } n \text{ is odd} \end{cases}$

for even n, the wavefunction is zero at $x = \frac{a}{2}$ at the location of the delta function. $\psi_n^{(0)}\Big|_{x=\frac{a}{2}} = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{2}\right) = 0$, for n even

so it never feels H'

b) $\psi_n^{(1)}(x) = \sum_{m \neq n} \frac{\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)}$;

now for G.S ; $n=1 \Rightarrow \psi_1^{(1)}(x) = \sum_{m \neq 1} \frac{\langle \psi_m^0 | H' | \psi_1^0 \rangle}{E_1^{(0)} - E_m^{(0)}} \psi_m^{(0)}$

where $\langle \psi_m^0 | H' | \psi_1^0 \rangle = \frac{2\alpha}{a} \int_0^a \sin\left(\frac{m\pi}{a}x\right) \delta(x-\frac{a}{2}) \sin\left(\frac{\pi}{a}x\right) dx$

$= \frac{2\alpha}{a} \sin\left(m\frac{\pi}{2}\right) = \begin{cases} 0 \,, & \text{for even } m \\ \frac{2\alpha}{a} \;; & \text{for odd } m \end{cases}$

so the first non zero terms are $m = 3, 5, 7$

with $E_1^{(0)} - E_m^{(0)} = \frac{\pi^2\hbar^2}{2ma^2}(1-m^2)$

$$\Rightarrow \psi_1^{(1)}(x) = \sum_{m=3,5,7,\dots} \frac{\frac{2\alpha}{a}\sin\left(m\frac{\pi}{2}\right)}{E_1^{(0)} - E_m^{(0)}} \psi_m^{(0)}$$

$$= \sum_{m=3,5,7} \frac{\frac{2\alpha}{a}\sin\left(m\frac{\pi}{2}\right)}{\frac{\pi^2\hbar^2}{2ma^2}(1-m^2)} \psi_m^{(0)}$$

$$= \frac{4\alpha m a}{\pi\hbar^2}\left\{ \frac{-1}{1-9}\psi_3^{(0)} + \frac{1}{1-25}\psi_5^{(0)} + \frac{-1}{1-49}\psi_7^{(0)} + \dots \right.$$

(c) $$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

where $\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle = \frac{2\alpha}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\delta\left(x-\frac{a}{2}\right)\sin\left(\frac{n\pi}{a}x\right)dx$

$$= \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{2\alpha}{a} & ; \text{ if both } m \text{ and } n \text{ are odd} \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Rightarrow E_n^{(2)} = \sum_{\substack{m\neq n \\ \text{odd}}} \left(\frac{2\alpha}{a}\right)^2 \frac{1}{E_n^{(0)} - E_m^{(0)}}$$

here we fix n and run over m with $m \neq n$

$$= \begin{cases} 0 & ; \text{ if } n \text{ is even} \\ 2m\left(\frac{2\alpha}{\pi\hbar}\right)^2 \sum\limits_{\substack{m\neq n \\ \text{odd}}} \frac{1}{n^2-m^2} & \end{cases}$$

mass

the last series $\sum_{m \neq n} \frac{1}{n^2 - m^2}$ can be summed for specific

n value ; $\sum_{\substack{m \neq n \\ odd}} \frac{1}{n^2 - m^2} = -\frac{1}{4n^2}$ [see Griffiths problems 6.1 and 6.4 (a)

$$\Rightarrow E_n^{(2)} = \begin{cases} 0, & \text{if } n \text{ is even} \\ -2m\left(\frac{\alpha}{\pi \hbar n}\right)^2 ; & \text{if } n \text{ is odd} \end{cases}$$

See that levels with even n are not affected by this perturbation

(2)
$$\left.\begin{aligned} \psi_{n_x}^{(0)}(x) &= \sqrt{\frac{2}{a}} \sin \frac{n_x \pi}{a} x \; ; \; n_x = 1, 2, 3, \ldots \\ \psi_{n_y}^{(0)}(y) &= \sqrt{\frac{2}{a}} \sin \frac{n_y \pi}{a} y \; ; \; n_y = 1, 2, 3, \ldots \end{aligned}\right\} \text{for unperturbed system}$$

$$\psi_{n_x, n_y}^{(0)} = \psi_{n_x}^{(0)} \psi_{n_y}^{(0)} = \frac{2}{a} \sin \frac{n_x \pi}{a} x \sin \frac{n_y \pi}{a} y$$

with $E_{n_x, n_y}^{(0)} = \frac{\hbar^2 \pi^2 n_x^2}{2ma^2} + \frac{\hbar^2 \pi^2 n_y^2}{2ma^2} = \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2)$

(a) for G.S $(n_x, n_y) = (1, 1)$

$$E_n^{(1)} = \langle n_x, n_y | H' | n_x, n_y \rangle = \langle 1,1 | H' | 1,1 \rangle$$

$$= \frac{4W_0}{a^2} \int_0^{a/2} \sin^2 \frac{\pi x}{a} \, dx \int_0^{a/2} \sin^2 \frac{\pi y}{a} \, dy = \frac{W_0}{4}$$

$$\therefore E_{1,1}^{(1)} = \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2) + \frac{W_0}{4} = \frac{\hbar^2 \pi^2}{ma^2} + \frac{W_0}{4}$$

(b) The first excited state is doubly degenerate

i.e $n_x = 1$ and $n_y = 2$ or $n_x = 2$ and $n_y = 1$

so need to the 2x2 matrix of H'

$$H' = \begin{pmatrix} \langle 1,2|w_0|1,2\rangle & \langle 1,2|w_0|2,1\rangle \\ \langle 2,1|w_0|1,2\rangle & \langle 2,1|w_0|2,1\rangle \end{pmatrix}$$

$$\text{so } \langle 1,2|w_0|1,2\rangle = w_0 \int_0^{a/2} \frac{a}{2}\sin^2\frac{\pi}{a}x\,dx \int_0^{a/2} \frac{a}{2}\sin^2\frac{2\pi y}{a}\,dy$$

$$= \frac{4w_0}{a^2}\int_0^{a/2}\sin^2\frac{\pi}{a}x\,dx\int_0^{a/2}\sin^2\frac{2\pi y}{a}\,dy$$

$$= \frac{4w_0}{a^2}\left[\frac{a}{4}\right]\left[\frac{a}{4}\right] = \frac{w_0}{4}$$

similarly we calculate the rest of the matrix elements

$$H' = w_0\begin{pmatrix} \frac{1}{4} & \frac{16}{9\pi^2} \\ \frac{16}{9\pi^2} & \frac{1}{4} \end{pmatrix} \Rightarrow \text{Find eigenvalues}$$

$$\begin{vmatrix} \frac{w_0}{4} - E & \frac{16w_0}{9\pi^2} \\ \frac{16w_0}{9\pi^2} & \frac{w_0}{4} - E \end{vmatrix} = 0 \Rightarrow \left(\frac{w_0}{4} - E\right)^2 - \left(\frac{16}{9\pi^2}w_0\right)^2 = 0$$

$$\frac{w_0}{4} - E = \pm\frac{16w_0}{9\pi^2}$$

$$E = \frac{w_0}{4} \pm \frac{16w_0}{9\pi^2}$$

So, to first order, the eigen energies are found by adding the eigenvalues of H' to the unperturbed eigen energies $\left(E^{(0)}_{1,2} = E^{(0)}_{2,1} = \frac{\hbar^2\pi^2(n_x^2+n_y^2)}{2ma^2}\right.$ $\left.= \frac{5\pi^2\hbar^2}{2ma^2}\right)$

$$\therefore E^{(1)} = \frac{5\pi^2\hbar^2}{2ma^2} + \frac{w_0}{4} \pm \frac{16w_0}{9\pi^2}$$

So degeneracy is lefted

③ for ideal 2D H.O, we have

$$H_0 |n_x, n_y\rangle = \hbar\omega (n_x + 1/2 + n_y + 1/2) |n_x, n_y\rangle = \hbar\omega (n_x + n_y + 1) |n_x, n_y\rangle$$

where $n_x = 0, 1, 2, \ldots$

$n_y = 0, 1, 2, \ldots$

$$\therefore \quad H_0 |n_x, n_y\rangle = \hbar\omega (n_x + n_y + 1) |n_x, n_y\rangle$$

now $|n_x, n_y\rangle = |n_x\rangle |n_y\rangle$

where $|n_x\rangle$ and $|n_y\rangle$ are the 1D eigenstates of the H.O

for example

$$\psi_0(x) = |0\rangle_x = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \quad \text{and} \quad \psi_0(y) = |0\rangle_y = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}y^2}$$

$$\psi_1(x) = |1\rangle_x = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar}\right)^{1/2} x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_1(y) = |1\rangle_y = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar}\right)^{1/2} y\, e^{-\frac{m\omega}{2\hbar}y^2}$$

so for the G.S of the 2D H.O, we have $n_x = n_y = 0$ and the corresponding eigenstate is (Non degenerate)

$$|0,0\rangle = |0\rangle_x |0\rangle_y = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

the energy correction to the G.S is

$$E_{0,0}(\beta) = \hbar\omega + \langle 0,0 | H' | 0,0\rangle + \sum_{n_x, n_y \neq 0,0} \frac{|\langle n_x, n_y | H' | 0,0\rangle|^2}{E^{(0)}_{0,0} - E^{(0)}_{n_x, n_y}}$$

where $H' = \beta m\omega^2 xy$

$$\langle 0,0 | \beta m\omega^2 xy | 0,0\rangle = \beta m\omega^2 \left(\frac{m\omega}{\pi\hbar}\right) \underbrace{\int_{-\infty}^{\infty} x e^{-\frac{m\omega}{2\hbar}x^2} dx}_{\text{odd}=0} \underbrace{\int_{-\infty}^{\infty} y e^{-\frac{m\omega}{2\hbar}y^2} dy}_{\text{odd}=0} = 0$$

$= 0$

so the second term does not contribute

now the third term reads

$$\sum_{n_x, n_y \neq 0,0} \frac{|\langle n_x, n_y | H' | 0,0 \rangle|^2}{E^{(0)}_{0,0} - E_{n_x, n_y}} \longrightarrow$$ to lowest order we take $n_x = 1, n_y = 1$

$$= \frac{|\langle 1,1 | H' | 0,0 \rangle|^2}{E^{(0)}_{0,0} - E^{(0)}_{1,1}}$$

now $\langle 1,1 | H' | 0,0 \rangle = \langle 1,1 | \beta m\omega^2 xy | 0,0 \rangle = \beta m\omega^2 \langle 1,1 | xy | 0,0 \rangle$

when $|1,1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2}$

and $$|0,0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}x^2} e^{-\frac{m\omega}{2\hbar}y^2}$$

$$\Rightarrow \langle 1,1 | H' | 0,0 \rangle = \beta m\omega^2 \left(\frac{m\omega}{\pi\hbar}\right) \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx \int_{-\infty}^{\infty} y^2 e^{-\frac{m\omega}{\hbar}y^2} dy$$

now using $\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \left(\frac{\pi}{4\alpha^3}\right)^{1/2}$; when $\alpha = \frac{m\omega}{\hbar}$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx = \left(\frac{\pi\hbar^3}{4m^3\omega^3}\right)^{1/2} \text{ and } \int_{-\infty}^{\infty} y^2 e^{-\frac{m\omega y^2}{\hbar}} dy = \left(\frac{\pi\hbar^3}{4m^3\omega^3}\right)^{1/2}$$

$$\Rightarrow \langle 1,1 | H' | 0,0 \rangle = \beta m\omega^2 \left(\frac{m\omega}{\pi\hbar}\right)\left(\frac{2m\omega}{\hbar}\right) \frac{\pi\hbar^3}{4m^3\omega^3} = \frac{1}{2}\hbar\omega\beta$$

$$|\langle 1,1 | H' | 0,0 \rangle|^2 = \frac{1}{4}\hbar^2\omega^2\beta^2$$

$$\Rightarrow \frac{|\langle 1,1 | H' | 0,0 \rangle|^2}{E^{(0)}_{0,0} - E^{(0)}_{1,1}} = \frac{\frac{1}{4}\hbar^2\omega^2\beta^2}{\hbar\omega - (3\hbar\omega)} = -\frac{1}{8}\hbar\omega\beta^2$$

$$\Rightarrow E^{(1)}_{0,0}(\beta) = \hbar\omega - \frac{1}{8}\hbar\omega\beta^2 = \hbar\omega\left(1 - \frac{\beta^2}{8}\right)$$

(b) the first excited state is doubly degenerate $(n_x, n_y) = (1,0)$ or $(0,1)$, so need to find the 2×2 H' matrix and its corresponding eigenvalues

$$H' = \begin{pmatrix} \langle 1,0|H'|1,0\rangle & \langle 1,0|H'|0,1\rangle \\ \langle 0,1|H'|1,0\rangle & \langle 0,1|H'|0,1\rangle \end{pmatrix}$$

where $|1,0\rangle = |1\rangle_x |0\rangle_y = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{2m\omega}{\hbar}\right)^{1/2} x\, e^{-\frac{m\omega}{2\hbar}x^2} e^{-\frac{m\omega}{2\hbar}y^2}$

and $|0,1\rangle = |0\rangle_x |1\rangle_y = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{2m\omega}{\hbar}\right)^{1/2} y\, e^{-\frac{m\omega}{2\hbar}x^2} e^{-\frac{m\omega}{2\hbar}y^2}$

so $\langle 1,0|H'|1,0\rangle = \langle 1,0|\beta m\omega^2 xy|1,0\rangle$

$$= \beta m\omega^2 \left(\frac{m\omega}{\pi\hbar}\right)\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{2\hbar}x^2}dx \underbrace{\int_{-\infty}^{\infty} y e^{-\frac{m\omega}{2\hbar}y^2}dy}_{\text{odd}=0}$$

$$= 0$$

similarly $\langle 0,1|H'|0,1\rangle = 0$

Now $\langle 1,0|H'|0,1\rangle = \beta m\omega^2 \langle 1,0|xy|0,1\rangle$

$$= \beta m\omega^2\left(\frac{m\omega}{\pi\hbar}\right)\left(\frac{2m\omega}{\hbar}\right)\underbrace{\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx}_{\left(\frac{\pi\hbar^3}{4m^3\omega^3}\right)^{1/2}} \underbrace{\int_{-\infty}^{\infty} y^2 e^{-\frac{m\omega}{\hbar}y^2}dy}_{\left(\frac{\pi\hbar^3}{4m^3\omega^3}\right)^{1/2}}$$

$$= \beta m\omega^2\left(\frac{m\omega}{\pi\hbar}\right)\left(\frac{2m\omega}{\hbar}\right)\left(\frac{\pi\hbar^3}{4m^3\omega^3}\right) =$$

$$= \frac{1}{2}\hbar\omega\beta = \langle 0,1|H'|1,0\rangle$$

$\therefore H' = \frac{1}{2}\hbar\omega\beta\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ need to find its eigenvalues

$$\begin{vmatrix} -E & \frac{1}{2}\hbar\omega\beta \\ \frac{1}{2}\hbar\omega\beta & -E \end{vmatrix} = 0 \quad \Rightarrow \quad E^2 - (\tfrac{1}{2}\hbar\omega\beta)^2 = 0$$

$$E = \pm \tfrac{1}{2}\hbar\omega\beta$$

so the energy of the first excited state to 1^{st} order

is $E^{(1)}_{1,0} = E^{(1)}_{0,1} = E^{(0)}_{1,0} \pm \frac{1}{2}\hbar\omega\beta$; $E^{(0)}_{1,0} = E^{(0)}_{0,1} = 2\hbar\omega$

$$= 2\hbar\omega \pm \tfrac{1}{2}\hbar\omega\beta$$

$$= 2\hbar\omega\left(1 \pm \frac{\beta}{4}\right)$$

so the degeneracy of the first excited state

is lefted

④ $H' = \beta(a^\dagger a^\dagger + aa)$ when $a|n\rangle = \sqrt{n}\,|n-1\rangle$

and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Now $E_n(\beta) = E_n^{(0)} + E_n^{(1)} + E_n^{(2)}$

$$= \hbar\omega(n+1/2) + \langle n|H'|n\rangle + \sum_{m\neq n}\frac{|\langle m|H'|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}} \quad \text{--- (1)}$$

$a^\dagger a^\dagger|n\rangle = \sqrt{n+1}\,a^\dagger|n+1\rangle = \sqrt{n+1}\sqrt{n+2}\,|n+2\rangle$

$aa|n\rangle = \sqrt{n}\,a|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle$

so $\langle m|H'|n\rangle = \beta\langle m|a^\dagger a^\dagger + aa|n\rangle$

$$= \beta\left[\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right] \quad \text{--- (2)}$$

$\Rightarrow \langle n|H'|n\rangle = 0$

so the only correction comes from the last term

$$\sum_{m\neq n}\frac{|\langle m|H'|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{|\langle n-2|H'|n\rangle|^2}{E_n^{(0)} - E_{n-2}^{(0)}} + \frac{|\langle n+2|H'|n\rangle|^2}{E_n^{(0)} - E_{n+2}^{(0)}} \quad \text{--- (3)}$$

as the only contribution to the sum comes only from the two terms $m = n-2$ and $m = n+2$ (see eq^n (2))

Now $E_n^{(0)} - E_{n-2}^{(0)} = \hbar\omega(n+1/2) - \hbar\omega(n-2+1/2) = +2\hbar\omega$

$E_n^{(0)} - E_{n+2}^{(0)} = \hbar\omega(n+1/2) - \hbar\omega(n+2+1/2) = -2\hbar\omega$

and $\langle n-2|H'|n\rangle = \beta\langle n-2|a^\dagger a^\dagger + aa|n\rangle = \beta\sqrt{n(n-1)}$

$\langle n+2|H'|n\rangle = \beta\langle n+2|a^\dagger a^\dagger + aa|n\rangle = \beta\sqrt{(n+1)(n+2)}$

so eq^n (3) becomes $\displaystyle\sum_{m\neq n} = \frac{\beta^2 n(n-1)}{+2\hbar\omega} + \frac{\beta^2(n+1)(n+2)}{-2\hbar\omega}$

$$\sum_{m \neq n} = \frac{\beta^2}{2\hbar\omega}\left[n(n-1) - (n+1)(n+2) \right]$$

$$= \frac{-2\beta^2}{\hbar\omega}(n+1/2)$$

$\Rightarrow$ equation (1) yields

$$E_n(\beta) = \hbar\omega(n+1/2) + 0 - \frac{2\beta^2}{\hbar\omega}(n+1/2)$$

$$= \hbar\omega(n+1/2)\left[1 - 2\left(\frac{\beta}{\hbar\omega}\right)^2\right]$$

(5) $\hat{H}_0 = \frac{L^2}{2I}$ and $\vec{\mu} = \alpha \vec{L}$, when α is a constant

$$\hat{H}' = -\vec{\mu}\cdot\vec{B} = -\alpha \vec{L}\cdot\vec{B} = -\alpha B L_z$$

for the unperturbed system, we have

$$\hat{H}_0 = \frac{\hat{L}^2}{2I} \quad ; \text{ where } \hat{L}^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \hat{H}_0 = \frac{\hbar^2}{I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

; we see that $\hat{H}_0$ is already diagonalized with three degenerate eigenvalues

$$E_1^{(0)} = E_2^{(0)} = E_3^{(0)} = \frac{\hbar^2}{I}$$

with the corresponding eigenstates

$$|\phi_1\rangle \; ; \; |\phi_2\rangle \; ; \; |\phi_3\rangle$$

$\Rightarrow$ for $E_1^{(0)} = \frac{\hbar^2}{I}$, we have

$$\frac{\hbar^2}{I}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{\hbar^2}{I}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \begin{matrix} a_1 = a_1 \\ a_2 = a_2 \\ a_3 = a_3 \end{matrix}$$

so we can not right now find the eigenstates of $\hat{H}_0$. but never mind, we know that $\hat{H}_0$ commute with L_z, meaning they $[\hat{H}_0, \hat{L}_z] = 0$, so $\hat{H}_0$ ~~will~~ have simultaneous eigenstates. therefore, we will find the eigenstates of $\hat{L}_z$ which will be the same eigenstates of $\hat{H}_0$.

now $H' = -\alpha B L_z$ with $L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$= -\alpha B \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

we see that H' is also diagonalized with non degenerate eigenvalues $-\alpha B\hbar$, 0, $+\alpha B\hbar$

$$\underset{E_3}{\downarrow} \qquad \underset{E_2}{\downarrow} \qquad \underset{E_1}{\downarrow}$$

let us find the eigenstates of H'

for E_1, we have $H_1 |a\rangle = E_1 |a\rangle$

$$+\alpha B\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = +\alpha B\hbar \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\Rightarrow a_1 = a_1 \ , \quad a_2 = 0 \ , \quad -a_3 = a_3 \Rightarrow a_3 = 0$$

$\therefore |a\rangle = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$ normalize $a_1^2 = 1 \Rightarrow a_1 = \pm 1$

take the positive

$$|a\rangle = |\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

similarly for $E_2 = 0 \Rightarrow |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

and for $E_3 = -\alpha B\hbar$, $|\phi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Notice that $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ are also an eigenstates of H_0 as $[H_0, H'] = [\frac{L^2}{2I}, -\alpha B L_z] = -\frac{\alpha B}{2I}[L^2, L_z] = 0$

Now let us find the energy correction to 1^{st} order

$$E_1^{(1)} = \langle \phi_1 | H' | \phi_1 \rangle = -\alpha B\hbar \, (1\ 0\ 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= -\alpha B\hbar \, (1 0 0) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\alpha B\hbar$$

$$E_2^{(1)} = \langle \phi_2 | H' | \phi_2 \rangle = -\alpha B\hbar \, (0\ 1\ 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

and finally

$$E_3^{(1)} = \langle \phi_3 | H' | \phi_3 \rangle = -\alpha B \hbar \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= +\alpha B \hbar$$

so the magnetic field has lefted the degeneracy to first order

- $E_1^{(0)} + E_1^{(1)} = \frac{\hbar^2}{I} - \alpha B \hbar$

- $E_2^{(0)} + E_2^{(1)} = \frac{\hbar^2}{I}$

- $E_3^{(0)} + E_3^{(1)} = \frac{\hbar^2}{I} + \alpha B \hbar$

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in general, we can write

$$E = \langle H \rangle = \frac{L^2}{2I} - \vec{\mu} \cdot \vec{B} = \frac{l(l+1)\hbar^2}{2I} - \alpha B \hbar m_l$$

with $-l \leq m_l \leq l$

in our case $l = 1 \Rightarrow m_l = -1, 0, +1$

$\Rightarrow$ Notice that we could have solved this problem in a shorter way

$H_0 = \frac{L^2}{2I} \Rightarrow \langle H_0 \rangle = E^{(0)} = \frac{\langle L^2 \rangle}{2I} = \frac{l(l+1)\hbar^2}{2I}$

and $H' = -\alpha B l_z \Rightarrow \langle H' \rangle = -\alpha B \langle l_z \rangle = -\alpha B \hbar m_l$

$\Rightarrow E = \frac{l(l+1)\hbar^2}{2I} - \alpha B \hbar m_l$; with $-l \leq m_l \leq l$