Graduate QM
HW # 3- solution
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(1)
$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\, G(\vec{r},\vec{r}')\, V(\vec{r}')\, \psi_k(\vec{r}')$$

let $\vec{r} \Rightarrow \vec{r}+\vec{R}$

$$\Rightarrow \psi_k(\vec{r}+\vec{R}) = e^{i\vec{k}\cdot(\vec{r}+\vec{R})} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\, G(\vec{r}+\vec{R},\vec{r}')\, V(\vec{r}')\, \psi_k(\vec{r}') \quad \text{------ (1)}$$

Now $G(\vec{r}) = \frac{1}{2\pi^2}\int \frac{1}{k'^2-k^2} e^{i\vec{k}\cdot\vec{r}} d^3\vec{k}'$

$$\Rightarrow G(\vec{r}+\vec{R}) = \frac{1}{2\pi^2}\int \frac{1}{k'^2-k^2} e^{i\vec{k}\cdot(\vec{r}+\vec{R})} d^3\vec{k}' = e^{i\vec{k}\cdot\vec{R}} \frac{1}{2\pi^2}\int \frac{1}{k'^2-k^2} e^{i\vec{k}\cdot\vec{r}} d^3\vec{k}'$$

$$= e^{i\vec{k}\cdot\vec{R}} G(\vec{r})$$

$$\Rightarrow G(\vec{r}+\vec{R},\vec{r}') = e^{i\vec{k}\cdot\vec{R}} G(\vec{r},\vec{r}')$$

inserting this in eq^n (1) yields

$$\psi_k(\vec{r}+\vec{R}) = e^{i\vec{k}\cdot\vec{R}}\left[e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\, G(\vec{r},\vec{r}')\, V(\vec{r}')\, \psi_k(\vec{r}')\right]$$

$$= e^{i\vec{k}\cdot\vec{R}}\psi_k(\vec{r}) \qquad \text{Q.E.D}$$

(b) $f(\vec{r}) = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}}\, V(\vec{r})$; $\vec{q} = \vec{k}-\vec{k}'$

$$f(\vec{r}+\vec{R}) = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\, e^{i\vec{q}\cdot(\vec{r}+\vec{R})}\, V(\vec{r}+\vec{R}) = e^{i\vec{q}\cdot\vec{R}} f(\vec{r})$$

for Periodic Potential and identical scatterers; $f(\vec{r})$ is also Periodic. this means $f(\vec{r}+\vec{R}) = f(\vec{r})$, indicating that $e^{i\vec{q}\cdot\vec{R}} = 1$

$\Rightarrow \cos(\vec{q}\cdot\vec{R}) = \cancel{2\pi n}\, 1 \Rightarrow \vec{q}\cdot\vec{R} = 2\pi n \Rightarrow (\vec{k}-\vec{k}')\cdot\vec{R} = 2\pi n$

Laue condition

② $$V(r) = \begin{cases} V_0 , & r<R \\ 0 , & r>R \end{cases}$$

ⓐ $$f(\theta) = -\frac{m}{2\pi\hbar^2} V_q \quad ; \quad V_q = \int d^3r \, e^{i\vec{q}\cdot\vec{r}} \, V(r)$$

$$\therefore \quad V_q = \frac{4\pi}{q}\int_0^R dr \, r \, V(r) \sin(qr) = \frac{4\pi}{q}\underbrace{\int_0^R dr \, r \, V_0 \sin qr}_{\text{integrate by parts}}$$

$$\Rightarrow f(\theta) = -\frac{m}{2\pi\hbar^2}\frac{4\pi}{q}V_0\left[-\frac{R}{q}\cos(qR) + \frac{1}{q^2}\sin qR\right]$$

$$= \frac{-2mV_0}{\hbar^2 q^3}\left[\sin(qR) - qR\cos(qR)\right]$$

where $q = 2k\sin\frac{\theta}{2}$

$u = r \quad , \quad dv = \sin qr$

$du = dr \, , \quad v = -\frac{\cos qr}{q}$

$$\int u\,dv = uv - \int v\,du$$

$$= -r\frac{\cos qr}{q} + \frac{1}{q}\int \cos qr \, dr$$

$$= \left[-r\frac{\cos qr}{q} + \frac{1}{q^2}\sin qr\right]_0^R$$

$$= -\frac{R}{q}\cos(qR) + \frac{1}{q^2}\sin qR$$

ⓑ low energy limit $qR \ll 1$

$$\sin qR \approx qR - \frac{1}{3!}(qR)^3$$

$$\cos qR = 1 - \frac{1}{2}(qR)^2$$

so $$f(\theta) = \frac{-2mV_0}{\hbar^2 q^3}\left[qR - \frac{1}{6}q^3R^3 - qR\left(1 - \frac{1}{2}q^2R^2\right)\right]$$

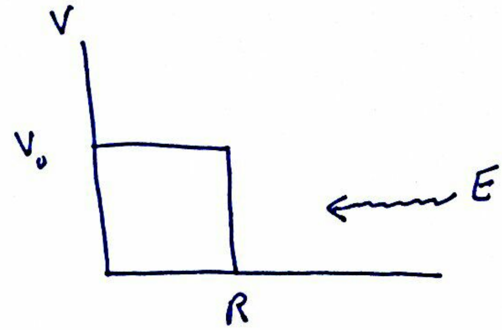
$$= -\frac{2mV_0}{\hbar^2 q^3}\left[-\frac{1}{6}q^3R^3 + \frac{1}{2}q^3R^3\right] = -\frac{2mV_0}{\hbar^2 q^3}\left[\frac{1}{3}q^3R^3\right]$$

$$= -\frac{2}{3}\frac{mV_0}{\hbar^2}R^3$$

$$\sigma = \frac{\pi}{k^2}\int_0^{4k^2} dw \, |f(q)|^2 = \frac{\pi}{k^2}\int_0^{4k^2} dw \left(\frac{2}{3}\frac{mV_0}{\hbar^2}R^3\right)^2 = \frac{\pi}{k^2}\left(\frac{2}{3}\frac{mV_0}{\hbar^2}R^3\right)^2\int_0^{4k^2} dw$$

$$= \frac{\pi}{k^2}\frac{4}{9}\frac{m^2V_0^2}{\hbar^4}R^6 \, 4k^2 = \frac{16\pi}{9}\frac{m^2V_0^2}{\hbar^4}R^6$$

(c) low energy limit (s-wave scattering ; $l=0$)



for $r>R$ $\quad \dfrac{d^2u_l}{dr^2} + \left[k^2 - \dfrac{l(l+1)}{r^2}\right]u_l = 0$

for $l=0 \Rightarrow \dfrac{d^2u_0}{dr^2} + k^2 u_0 = 0 \;;\; k^2 = \dfrac{2mE}{\hbar^2}$

$$\boxed{u_0 = A\sin(kr+\delta_0)} \quad \text{---- (1)}$$

for $r<R$ $\quad \dfrac{d^2u_l}{dr^2} + \left[k'^2 - \dfrac{l(l+1)}{r^2}\right]u_l = 0$

for $l=0 \quad \dfrac{d^2u_0}{dr^2} + k'^2 u_0 = 0 \quad ; \quad k'^2 = \dfrac{2m}{\hbar^2}(E-V_0)$

but $E<V_0 \Rightarrow k'$ is imaginary

$u_0 = A\cosh(k'r) + B\sinh(k'r)$

u_0 has to vanish at $r=0 \quad \Rightarrow A=0$

$$\Rightarrow \boxed{u_0 = B\sin(k'r)} \quad \text{---- (2)}$$

Now matching the log derivative at $r=R$, yields

$$\left.\frac{u_0'}{u_0}\right|_{r<R} = \left.\frac{u_0'}{u_0}\right|_{r>R} \Rightarrow \frac{k'\cosh k'R}{\sinh k'R} = \frac{k\cos(ka+\delta_0)}{\sin(ka+\delta_0)}$$

$$\Rightarrow \frac{1}{k'}\tanh(k'R) = \frac{1}{k}\tan(ka+\delta_0)$$

$$\frac{k}{k'}\tanh(k'R) = \tan(ka+\delta_0) \quad \text{---} \quad (3)$$

Now for low energy limit $\tan(ka+\delta_0) \simeq ka+\delta_0$

and $\tanh(k'R) = k'R - \frac{1}{3}k'^3R^3$; using $\tanh x = x - \frac{1}{3}x^3$ for $x \ll 1$

inserting these limits in (3), gives

$$\frac{k}{k'}\left(k'R - \frac{1}{3}k'^3R^3\right) = kR + \delta_0$$

$$\cancel{kR} - \frac{1}{3}k'^2R^3 = \cancel{kR} + \delta_0 \quad \Rightarrow \quad \delta_0 = -\frac{1}{3}kk'^2R^3$$

Now the scattering amplitude f is

$$f = \frac{1}{k}e^{i\delta_0}\sin\delta_0 \quad \Rightarrow \quad |f|^2 = \frac{1}{k^2}\sin^2\delta_0 \approx \frac{1}{k^2}\delta_0^2$$

$$|f|^2 = \frac{1}{k^2}\frac{1}{9}k^2k'^4R^6 = \frac{1}{9}k'^4R^6 = \frac{1}{9}\frac{4m^2}{\hbar^4}(E-V_0)^2R^6$$

but $E \ll V_0$

$$|f|^2 = \frac{1}{9}\frac{4m^2}{\hbar^4}V_0^2R^6 \quad ; \quad |f| = \frac{2}{3}\frac{mV_0}{\hbar^2}R^3 \quad \text{same as part b)}$$

Now $\sigma_T = \int \frac{d\sigma}{d\Omega}d\Omega \quad ; \quad \frac{d\sigma}{d\Omega} = |f|^2$

$$= \frac{4}{9}\frac{m^2V_0^2}{\hbar^4}R^6(4\pi) = \frac{16\pi}{9}\frac{m^2V_0^2}{\hbar^4}R^6 \quad \text{same as part b)}$$

-for very high potential $(V_0 \to \infty) \Rightarrow k'R \to \infty$

$\therefore$ from (3) $\tan(kR+\delta_0) = \frac{k}{k'}\tanh(k'R) \quad ; \quad \tanh(k'R) \to 1$

$$\tan(kR+\delta_0) \simeq \frac{k}{k'}$$

but $k' \gg k$ or $\frac{k}{k'} \to 0$

$$\Rightarrow \tan(kR+\delta_0) \to 0 \Rightarrow kR + \delta_0 = 0 \Rightarrow \boxed{\delta_0 = -kR}$$

$$\sigma_T = \frac{4\pi}{k^2}\delta_0^2 = \frac{4\pi}{k^2}(-kR)^2 = \underline{\underline{4\pi R^2}}$$

hard sphere scattering as expected

③ $V(r) = \frac{g}{r} e^{-\mu r}$; Yukawa Potential

ⓐ $f = \frac{-m}{2\pi\hbar^2} V_q$;

$$V_q = \int d^3r\, e^{i\vec{q}\cdot\vec{r}} V(r) = \int_0^\infty r^2 dr \frac{g}{r} e^{-\mu r} \underbrace{\int_0^\pi e^{iqr\cos\alpha} \sin\alpha\, d\alpha}_{\frac{1}{iqr}(e^{iqr} - e^{-iqr})} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{2\pi g}{iq} \int_0^\infty dr \left(e^{-(\mu - iq)r} - e^{-(\mu + iq)r}\right)$$

$$= \frac{2\pi g}{iq}\left[\frac{1}{\mu - iq} - \frac{1}{\mu + iq}\right] = \frac{2\pi g}{iq} \frac{2iq}{\mu^2 + q^2} = \frac{4\pi g}{\mu^2 + q^2}$$

$$\Rightarrow f = -\frac{2mg}{\hbar^2} \frac{1}{(\mu^2 + q^2)} \Rightarrow |f|^2 = \frac{4m^2 g^2}{\hbar^4} \frac{1}{(\mu^2 + q^2)^2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f|^2 = \frac{4m^2 g^2}{\hbar^4} \frac{1}{(\mu^2 + q^2)^2}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4m^2 g^2}{\hbar^4} \int d\Omega \frac{1}{(\mu^2 + q^2)^2} = \frac{4m^2 g^2}{\hbar^4} \int_0^\pi \frac{\sin\theta\, d\theta}{(\mu^2 + q^2)^2} \int_0^{2\pi} d\phi$$

$$= \frac{8\pi m^2 g^2}{\hbar^4} \int_0^\pi \frac{\sin\theta\, d\theta}{(\mu^2 + q^2)^2} = \lambda \int_0^\pi \frac{\sin\theta\, d\theta}{(\mu^2 + q^2)^2} \;;\; \text{where } \lambda = \frac{8\pi m^2 g^2}{\hbar^4}$$

let $w = q^2 = 4k^2 \sin^2\theta/2 \Rightarrow dw = 2k^2 \sin\theta\, d\theta$

$$\Rightarrow \sigma = \frac{\lambda}{2k^2} \int_0^{4k^2} \frac{dw}{(\mu^2 + w)^2} \;;\; \text{let } u = \mu^2 + w,\; du = dw$$

$$= \frac{\lambda}{2k^2} \int_{\mu^2}^{\mu^2 + 4k^2} \frac{du}{u^2} = -\frac{\lambda}{2k^2}\left[\frac{1}{u}\right]_{\mu^2}^{\mu^2 + 4k^2} = \frac{2\lambda}{\mu^2(\mu^2 + 4k^2)}$$

------ (1)

(b) low energy limit

$$V_q = \int d^3 r \, e^{i\vec{q}\cdot\vec{r}} \, V(r) \; ; \quad e^{i\vec{q}\cdot\vec{r}} \to 1$$

$$= \int_0^\infty dr \, r^2 \frac{g}{r} e^{-\mu r} \int_0^{4\pi} d\Omega = 4\pi g \int_0^\infty dr \, r \, e^{-\mu r}$$

$$= \frac{4\pi g}{\mu^2}$$

$$\Rightarrow f = -\frac{m}{2\pi\hbar^2} V_q = -\frac{2mg}{\hbar^2 \mu^2} \Rightarrow |f|^2 = \frac{4m^2 g^2}{\hbar^4 \mu^4}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f|^2$$

$$\sigma = \int |f|^2 d\Omega = \frac{16\pi m^2 g^2}{\hbar^4 \mu^4}$$

same result can be obtained from result of part (a) eq^n (1) by letting $k \to 0$

(c) high energy limit

using eq^n (1) ; $|f|^2 = \frac{4m^2 g^2}{\hbar^4} \frac{1}{(\mu^2+q^2)^2}$

$$\sigma = \int |f|^2 d\Omega = \frac{4m^2 g^2}{\hbar^4} \int d\Omega \frac{1}{(\mu^2+q^2)^2} = \frac{8\pi m^2 g^2}{\hbar^4} \int_0^\pi \frac{\sin\theta \, d\theta}{(\mu^2+q^2)^2}$$

again let $w = q^2$ (see part (a))

$$\sigma = \frac{\lambda}{2k^2} \int_0^{4k^2} \frac{dw}{(\mu^2+w)^2} \; ; \text{ when } \lambda = \frac{8\pi m^2 g^2}{\hbar^4}$$

for high energy limit $k \to \infty \Rightarrow \sigma = \frac{\lambda}{2k^2} \int_0^\infty \frac{dw}{(\mu^2+w)^2}$

let $u = \mu^2 + w \Rightarrow du = dw$

$$\Rightarrow \sigma = \frac{\lambda}{2k^2} \int_{\mu^2}^\infty \frac{du}{u^2} = \frac{\lambda}{2k^2 \mu^2}$$

; same result can be obtained from eq^n (1) by letting $k \to \infty$ $(\mu^2 + 4k^2) \to 4k^2$

④ $V(r) = -V_0 e^{-r/a}$, where $V_0 > 0$ and a is a constant

a) $\frac{d\sigma}{d\Omega} = |f|^2$; $f = \frac{-m}{2\pi\hbar^2} V_q$

$$V_q = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)$$

$$= -V_0 \int_0^\infty dr\, r^2 e^{-r/a} \underbrace{\int_0^\pi e^{iqr\cos\alpha} \sin\alpha\, d\alpha}_{\frac{1}{iqr}(e^{iqr} - e^{-iqr})} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= -\frac{2\pi V_0}{iq}\left[\int_0^\infty r\,dr \left[e^{-(\frac{1}{a}-iq)r} - e^{-(\frac{1}{a}+iq)r}\right]\right]$$

Now using $\int_0^\infty r^n e^{-\beta r} dr = \frac{n!}{(\beta)^{n+1}}$

$$= \frac{-2\pi V_0}{iq}\left[\frac{1}{(\frac{1}{a}-iq)^2} - \frac{1}{(\frac{1}{a}+iq)^2}\right]$$

$$= -\frac{2\pi V_0}{iq}\left[\frac{a^2}{(1-iqa)^2} - \frac{a^2}{(1+iqa)^2}\right] = -\frac{2\pi V_0 a^2}{iq}\left[\frac{(1+iqa)^2 - (1-iqa)^2}{(1+q^2a^2)^2}\right]$$

$$= \frac{-2\pi V_0 a^2}{iq}\left[\frac{1+2iqa+(iqa)^2 - (1-2iqa+(iqa)^2)}{(1+q^2a^2)^2}\right]$$

$$= \frac{-2\pi V_0 a^2}{iq}\left[\frac{4iqa}{(1+q^2a^2)^2}\right] = -\frac{8\pi V_0 a^3}{(1+q^2a^2)^2}$$

$$\Rightarrow f = \frac{-m}{2\pi\hbar^2} V_q = \frac{-m}{2\pi\hbar^2}\left(-\frac{8\pi V_0 a^3}{(1+q^2a^2)^2}\right) = \frac{4mV_0 a^3}{\hbar^2}\frac{1}{(1+q^2a^2)^2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f|^2 = \frac{16\, m^2 V_0^2 a^6}{\hbar^4} \frac{1}{(1+q^2a^2)^4} \quad ; \quad \text{---} \quad (1)$$

$$= \frac{\lambda}{(1+q^2a^2)^4} \quad ; \text{ when } \lambda = \frac{16 m^2 V_0^2 a^6}{\hbar^4}$$

(b) low energy limit $qa \ll 1$

two methods:

1) method 1

$V_q = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, v(r)$; for low energy $e^{i\vec{q}\cdot\vec{r}} \longrightarrow 1$

$$= \int d^3r\, v(r) = \int_0^\infty r^2 dr\, (-V_0 e^{-r/a}) \int d\Omega$$

$$= -4\pi V_0 \int_0^\infty r^2 e^{-r/a} dr = -4\pi V_0 \frac{2}{(1/a)^3} = -8\pi V_0 a^3$$

$$\Rightarrow f = \frac{-m}{2\pi\hbar^2} V_q = \frac{4 m V_0 a^3}{\hbar^2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f|^2 = \frac{16\, m^2 V_0^2 a^6}{\hbar^4}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{64\pi m^2 V_0^2 a^6}{\hbar^4} \equiv \text{constant}$$

2) method 2

the scattering can be read directly from result of part (a) eq^n (1) by letting $qa \to 0$

$$\Rightarrow |f|^2 = \frac{d\sigma}{d\Omega} = \frac{16\, m^2 V_0^2 a^6}{\hbar^4} \quad \Rightarrow \sigma = \frac{64\pi m^2 V_0^2 a^6}{\hbar^4}$$

(c) high energy limit $qa >> 1$

$$\sigma = \int |f|^2 d\Omega = \int \frac{\lambda}{(1+q^2a^2)^4} \sin\theta \, d\theta \, d\phi$$

$$= 2\pi\lambda \int_0^{\pi} \frac{\sin\theta \, d\theta}{(1+q^2a^2)^4} \quad ;$$

let $w = a^2q^2 = 4k^2a^2 \sin^2\frac{\theta}{2}$; when $q = 2k \sin\frac{\theta}{2}$

$$dw = 8k^2a^2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cdot \frac{1}{2} d\theta = 4k^2a^2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}$$

using $\sin 2x = 2\sin x \cos x$

$$= 4k^2a^2 \frac{1}{2} \sin\theta \, d\theta$$

$$= 2k^2a^2 \sin\theta \, d\theta$$

$$\Rightarrow \sigma = \frac{2\pi\lambda}{2k^2a^2} \int_0^{4k^2a^2} \frac{dw}{(1+w)^4} = \frac{2\pi\lambda}{2k^2a^2} \int_0^{\infty} (1+w)^{-4} dw$$

when $4k^2a^2 \to \infty$ for high energy limit

$$= -\frac{\pi\lambda}{3k^2a^2} \left[\frac{1}{(1+w)^3} \right]_0^{\infty} = \frac{\pi\lambda}{3a^2} \frac{1}{k^2}$$

$$= \frac{\pi}{3a^2} \frac{16 m^2 V_0^2 a^6}{\hbar^4} \frac{1}{k^2}$$

$$= \frac{16\pi m^2 V_0^2 a^4}{3\hbar^4} \frac{1}{k^2}$$

⑤ $V(r) = V_0\,\delta(r-a)$;

(a) $f = \frac{-m}{2\pi\hbar^2} V_q$; $V_q = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)$; $\frac{d\sigma}{d\Omega} = |f|^2$

$$V_q = V_0 \int_0^\infty r^2 dr\, \delta(r-a) \underbrace{\int_0^\pi e^{iqr\cos\alpha} \sin\alpha\, d\alpha}_{\frac{1}{iqr}(e^{iqr} - e^{-iqr})} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= 2\pi V_0 \int_0^\infty r^2 dr\, \delta(r-a) \frac{1}{iqr}(e^{iqr} - e^{-iqr})$$

$$= \frac{4\pi V_0}{q} \int_0^\infty dr\; r\; \delta(r-a) \sin qr \;=\; \frac{4\pi V_0}{q}\, a \sin qa$$

$$\Rightarrow f = \frac{-m}{2\pi\hbar^2} V_q = \frac{-m}{2\pi\hbar^2}\, \frac{4\pi V_0}{q}\, a\sin qa = \frac{-2mV_0}{\hbar^2 q}\, a\sin qa$$

(b) for low energy limit $\sin qa \approx qa$

$$\Rightarrow f = \frac{-2mV_0}{\hbar^2 q}\, a(qa) = -\frac{2mV_0}{\hbar^2} a^2$$

$$|f|^2 = \frac{4m^2V_0^2}{\hbar^4} a^4 = \frac{d\sigma}{d\Omega}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4m^2V_0^2 a^4}{\hbar^4} \int_0^{4\pi} d\Omega = \frac{4m^2V_0^2 a^4}{\hbar^4} 4\pi$$

$$= \frac{16\pi m^2 V_0^2}{\hbar^4} a^4$$

6 $V(r) = A e^{-\mu r^2}$

a

$$\frac{d\sigma}{d\Omega} = |f|^2 \;;\; f = -\frac{m}{2\pi\hbar^2} V_q$$

$$V_q = \int d^3r \; e^{i\vec{q}.\vec{r}} V(r)$$

$$= A \int r^2 dr \, e^{-\mu r^2} \underbrace{\int_0^{\pi} e^{iqr\cos\alpha} \sin\alpha \, d\alpha}_{\frac{1}{iqr}(e^{iqr} - e^{-iqr})} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{2\pi A}{iq} \int_0^{\infty} dr \; r e^{-\mu r^2} (e^{iqr} - e^{-iqr})$$

$$= \frac{2\pi A}{iq} \int_0^{\infty} dr \; r \left[e^{-\mu r^2 + iqr} - e^{-\mu r^2 - iqr} \right]$$

completing the squares

$$e^{-\mu r^2 + iqr} = e^{-\mu (r^2 - \frac{iq}{\mu} r)} = e^{-\mu (r^2 - \frac{iq}{\mu} r - \frac{q^2}{4\mu^2} + \frac{q^2}{4\mu^2})} = e^{-\frac{q^2}{4\mu}} e^{-\mu (r - \frac{iq}{2\mu})^2}$$

similarly $e^{-\mu r^2 - iqr} = e^{-\frac{q^2}{4\mu}} e^{-\mu (r + \frac{iq}{2\mu})^2}$

$$\Rightarrow V_q = \frac{2\pi A}{iq} e^{-\frac{q^2}{4\mu}} \left[\underbrace{\int_0^{\infty} dr \; r e^{-\mu (r - \frac{iq}{2\mu})^2}} - \int_0^{\infty} dr \; r \, e^{-\mu (r + \frac{iq}{2\mu})^2} \right]$$

let $u = r - \frac{iq}{2\mu} \Rightarrow du = dr$

$$\int_0^{\infty} du \left(u + \frac{iq}{2\mu}\right) e^{-\mu u^2} = \underbrace{\int_0^{\infty} du \; u \, e^{-\mu u^2}}_{\frac{1}{2\mu}} + \frac{iq}{2\mu} \underbrace{\int_0^{\infty} du \; e^{-\mu u^2}}_{\frac{1}{2}\sqrt{\frac{\pi}{\mu}}}$$

where I used

$$\int_0^\infty x e^{-ax^2} dx = \frac{1}{2a} \quad \text{and} \quad \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$\Rightarrow$ in general let $I_0 = \int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$

and $I_1 = \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}$

so $\int_0^\infty x^{2n} e^{-\alpha x^2} dx = I_{2n} = (-1)^n \frac{d^n}{d\alpha^n} I_0$

and $\int_0^\infty x^{2n+1} e^{-\alpha x^2} dx = I_{2n+1} = (-1)^n \frac{d^n}{d\alpha^n} I_1$

Gaussian integrals

$$\therefore V_q = \frac{2\pi A}{iq} e^{-\frac{q^2}{4\mu}} \left[\left(\frac{1}{2\mu} + \frac{iq}{2\mu}\frac{1}{2}\sqrt{\frac{\pi}{\mu}}\right) - \left(\frac{1}{2\mu} - \frac{iq}{2\mu}\frac{1}{2}\sqrt{\frac{\pi}{\mu}}\right)\right]$$

$$= \frac{2\pi A}{iq} e^{-\frac{q^2}{4\mu}} \left[\frac{iq}{2\mu}\sqrt{\frac{\pi}{\mu}}\right] = \frac{\pi A}{\mu}\sqrt{\frac{\pi}{\mu}} e^{-\frac{q^2}{4\mu}}$$

$$\Rightarrow f = -\frac{m}{2\pi\hbar^2} V_q = \frac{-mA\sqrt{\pi}}{2\hbar^2 \mu^{3/2}} e^{-q^2/4\mu}$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{m^2 A^2 \pi}{4\hbar^4 \mu^3} e^{-\frac{q^2}{2\mu}}$$

$$\sigma = \int |f|^2 d\Omega = \frac{m^2 A^2 \pi}{4\hbar^4 \mu^3} \int d\Omega\, e^{-\frac{q^2}{2\mu}} = \frac{m^2 A^2 \pi}{4\hbar^4 \mu^3} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta\, e^{-\frac{q^2}{2\mu}}$$

$$= \frac{m^2 A^2 \pi^2}{2\hbar^4 \mu^3} \int_0^\pi \sin\theta\, d\theta\, e^{-\frac{q^2}{2\mu}} ;$$

let $\omega = q^2 = 4k^2 \sin^2\frac{\theta}{2}$; $d\omega = 2k^2 \sin\theta\, d\theta$

$$\Rightarrow \sigma = \frac{m^2 A^2 \pi^2}{2\hbar^4 M^3} \int_0^{4k^2} \frac{dw}{2k^2} e^{-\frac{w}{2M}} = \frac{m^2 A^2 \pi^2}{4\hbar^4 M^3} \frac{1}{k^2} \left[\frac{e^{-w/2M}}{(-1/2M)} \right]_0^{4k^2}$$

$$= \frac{-m^2 A^2 \pi^2}{2\hbar^4 M^2 k^2} \left\{ e^{-\frac{2k^2}{M}} - 1 \right\}$$

$$= \frac{m^2 A^2 \pi^2}{2\hbar^4 M^2 k^2} \left\{ 1 - e^{-\frac{2k^2}{M}} \right\}$$

(b) - low energy limit $k \ll 1$; $e^{-\frac{2k^2}{M}} \approx 1 - \frac{2k^2}{M}$

$$\Rightarrow \sigma = \frac{m^2 A^2 \pi^2}{2\hbar^4 M^2 k^2} \left\{ \cancel{1} - \cancel{1} + \frac{2k^2}{M} \right\} = \frac{m^2 A^2 \pi^2}{2\hbar^4 M^2 k^2} \frac{2k^2}{M}$$

$$= \frac{m^2 A^2 \pi^2}{\hbar^4 M^3} \equiv \text{Constant}$$

(c) - high energy limit $k \to \infty$; $e^{-\frac{2k^2}{M}} \to 0$

$$\Rightarrow \sigma = \frac{m^2 A^2 \pi^2}{2\hbar^4 M^2} \frac{1}{k^2} \sim \frac{1}{k^2}$$

(7) $V(r) = \frac{A}{r^2}$; A is a constant

$$\frac{d\sigma}{d\Omega} = |f|^2 ; \quad f = -\frac{m}{2\pi\hbar^2} V_q$$

$$V_q = \int d^3r \, e^{i\vec{q}\cdot\vec{r}} V(r) = \int_0^\infty r^2 dr \, V(r) \underbrace{\int_0^\pi e^{iqr\cos\alpha} \sin\alpha \, d\alpha}_{\frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{2\pi A}{iqr} \int_0^\infty 2i \sin qr \, dr$$

$$= 4\pi A \int_0^\infty \frac{\sin qr}{qr} dr \quad ; \quad \text{let } x = qr, \ dx = q\,dr$$

$$= \frac{4\pi A}{q} \underbrace{\int_0^\infty \frac{\sin x}{x} dx}_{\pi/2} = \frac{2\pi^2 A}{q}$$

$$\Rightarrow f = -\frac{m}{2\pi\hbar^2}\left(\frac{2\pi^2 A}{q}\right) = -\frac{m\pi A}{\hbar^2 q}$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{m^2\pi^2A^2}{\hbar^4 q^2}$$

$$\sigma = \int |f|^2 d\Omega = \frac{m^2\pi^2A^2}{\hbar^4} \int \frac{1}{q^2} d\Omega$$

$$= \frac{m^2\pi^2A^2}{\hbar^4} \int_0^{2\pi} d\phi \int_0^\pi \frac{\sin\theta \, d\theta}{q^2}$$

$$= \frac{2m^2\pi^2A^2}{\hbar^4} \int_0^\pi \frac{\sin\theta \, d\theta}{q^2}$$

let $w = q^2 = 4k^2 \sin^2\frac{\theta}{2}$

$dw = 2k^2 \sin\theta \, d\theta$

$$\sigma = \frac{2m^2\pi^2A^2}{\hbar^4}\int_0^{4k^2}\frac{dw}{2k^2}\,\frac{1}{w} = \frac{m^2\pi^2A^2}{\hbar^4 k^2}\int_0^{4k^2} w^{-1}\,dw$$

$$= \frac{m^2\pi^2A^2}{\hbar^4 k^2}\Big[\ln w\Big]_0^{4k^2} \quad ?$$

singular at $w=0$

so σ diverges for this potential that is the Born approximation is <u>Not</u> applicable for this potential

⑧ $V(r) = \frac{A}{r}$; Coulomb Potential

$$\frac{d\sigma}{d\Omega} = |f|^2 \ ; \ f = \frac{-m}{2\pi\hbar^2} V_q$$

$$V_q = \int d^3r \, e^{i\vec{q}\cdot\vec{r}} \, V(r)$$

$$= \int_0^\infty r^2 dr \, \frac{A}{r} \int_0^\pi e^{iqr\cos\alpha} \sin\alpha \, d\alpha \int_0^{2\pi} d\phi$$

$$= \frac{2\pi A}{iq} \int_0^\infty (e^{iqr} - e^{-iqr}) dr$$

$$= \frac{2\pi A}{iq} \left[\frac{e^{iqr}}{iq} - \frac{e^{-iqr}}{-iq} \right]_0^\infty$$ diverges at $r = \infty$

therefore, we seek a modified short range Potential like the Yukawa Potential $V(r) = \frac{A}{r} e^{-\mu r}$, and then take the limit as $\mu \to 0$ to recover the Coulomb Potential $V(r) = \frac{A}{r}$

so from Yukawa Potential (See problem 3), we found

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{4m^2A^2}{\hbar^4} \frac{1}{(\mu^2+q^2)^2}$$

so let $\mu \to 0 \Rightarrow |f|^2 = \frac{4m^2A^2}{\hbar^4} \frac{1}{q^4}$

when $q^2 = 4k^2 \sin^2\frac{\theta}{2}$ and $A = Z_1 Z_2 e^2$

Z_1: charge of scatterer

Z_2: charge of incident particle

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f|^2 = \frac{4m^2A^2}{\hbar^4} \frac{1}{16k^4 \sin^4(\frac{\theta}{2})} \quad \text{Converges}$$

now $\sigma_T = \int |f|^2 d\Omega = \frac{4m^2A^2}{\hbar^4} \int \frac{d\Omega}{q^4}$

$$= \frac{4m^2A^2}{\hbar^4} \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\sin\theta \, d\theta}{q^4} = \frac{8\pi m^2A^2}{\hbar^4} \int_0^{\pi} \frac{\sin\theta \, d\theta}{q^4}$$

$$= \frac{8\pi m^2A^2}{\hbar^4 (2k^2)} \int_0^{4k^2} \frac{dw}{w^2}$$

$$= \frac{4\pi m^2A^2}{\hbar^4 k^2} \left[-\frac{1}{w} \right]_0^{4k^2} \quad \text{diverges at } w=0$$

so although the differential cross section converges, the total cross section diverges. this is due to the coulomb potential is long range potential. that the incident particles are so no matter how far the incident particles are from the target charge, there is always an effect on the motion of the incident particles and hence they get scattered.