# Graduate QM

## HW #2 - Solution

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① ⓐ $[L_i, \frac{p^2}{2m}] = [(r\times p)_i, \frac{p^2}{2m}] = [\varepsilon_{ijk} x_j p_k, \frac{p^2}{2m}]$

$$= \frac{\varepsilon_{ijk}}{2m} [x_j p_k, p^2]$$

$$= \frac{\varepsilon_{ijk}}{2m} \left( x_j \underbrace{[p_k, p^2]}_{\to 0} + [x_j, p^2] p_k \right)$$

$$= \frac{\varepsilon_{ijk}}{2m} [x_j, p_j^2] p_k \quad ; \text{ when } [x_j, p_i^2] = [x_j, p_k^2] = 0$$

$$= \frac{\varepsilon_{ijk}}{2m} i\hbar \frac{\partial p_j^2}{\partial p_j} p_k = \frac{\varepsilon_{ijk}}{2m} i\hbar\, 2 p_j p_k$$

$$= \frac{i\hbar}{m} \varepsilon_{ijk} p_j p_k = \frac{i\hbar}{m} (p\times p)_i = 0 \qquad \text{Q.E.D}$$

where I used $(a\times b)_i = \varepsilon_{ijk} a_j b_k$

ⓑ starting from $i\hbar \frac{d}{dt} \langle L_i \rangle = \langle [L_i, H] \rangle$

where $H = \frac{p^2}{2m} + V(\vec{r})$

$$\Rightarrow i\hbar \frac{d}{dt} \langle L_i \rangle = \left\langle \left[ (r\times p)_i, \frac{p^2}{2m} + V(\vec{r}) \right] \right\rangle = \left\langle \left[ \varepsilon_{ijk} x_j p_k, \frac{p^2}{2m} + V(\vec{r}) \right] \right\rangle$$

$$= \varepsilon_{ijk} \left\langle [x_j p_k, \frac{p^2}{2m}] + [x_j p_k, V(\vec{r})] \right]$$

$$i\hbar \frac{d}{dt}\langle L_i \rangle = \varepsilon_{ijk} \Big\langle \frac{1}{2m} x_j [\cancel{P_k, P^2}]^{\to 0} + \frac{1}{2m}[x_j, P^2] P_k$$

$$+ x_j [P_k, V(\vec{r})] + [\cancel{x_j, V(\vec{r})}]^{\to 0} P_k \Big\rangle$$

$$= \varepsilon_{ijk} \Big\langle \frac{1}{2m} 2i\hbar P_j P_k + x_j \Big(-i\hbar \frac{\partial V}{\partial x_k}\Big) \Big\rangle$$

$$= \frac{1}{m} i\hbar\, \varepsilon_{ijk} P_j P_k - \varepsilon_{ijk} x_j\, i\hbar\, \partial_k V \quad ; \text{where } \frac{\partial V}{\partial x_k} = \partial_k V \text{ shortly}$$

$$= \frac{1}{m} i\hbar \cancel{(P \times P)_i}^{\to 0} - i\hbar\, \varepsilon_{ijk} x_j \partial_k V$$

$$= -i\hbar\, \varepsilon_{ijk} x_j \partial_k V = -i\hbar (\vec{r} \times \nabla V)_i$$

$$\therefore \frac{d}{dt}\langle L_i \rangle = -(\vec{r} \times \vec{\nabla} V)_i$$

$$\Rightarrow \frac{d}{dt}\langle \vec{L} \rangle = -(\vec{r} \times \vec{\nabla} V) \qquad \text{Q.E.D}$$

(c) $$[L_i, f(p^2)] = [(r \times P)_i, f(p^2)] = [\varepsilon_{ijk} x_j P_k, f(p^2)]$$

$$= \varepsilon_{ijk} \Big( x_j [\cancel{P_k, f(p^2)}]^{\to 0} + [x_j, f(p^2)] P_k \Big) = \varepsilon_{ijk} [x_j, f(p^2)] P_k$$

$$= \varepsilon_{ijk}\, i\hbar \frac{\partial f(P_j^2)}{\partial P_j} P_k = (\text{const})\, i\hbar\, \varepsilon_{ijk} P_j P_k$$

$$= (\text{const})\, i\hbar (P \times P)_i = 0$$

$\Rightarrow$

similarly $[L_j, f(p^2)] = [L_k, f(p^2)] = 0$

$$\Rightarrow [\vec{L}, f(p^2)] = 0$$

$$[L_i, f(r^2)] = [(r\times p)_i, f(r^2)]$$

$$= [\varepsilon_{ijk} x_j p_k, f(r^2)]$$

$$= \varepsilon_{ijk}\left(x_j [p_k, f(r^2)] + \underbrace{[x_j, f(r^2)]}_{\to 0} p_k\right)$$

$$= \varepsilon_{ijk} x_j [p_k, f(r^2)]$$

$$= \varepsilon_{ijk} x_j [p_k, f(x_k^2)] \quad ; \text{ where } [p_k, f(x_i^2)] = 0, \quad [p_k, f(x_j^2)] = 0$$

$$= \varepsilon_{ijk} x_j \left(-i\hbar \frac{\partial f}{\partial x_k}\right)$$

$$= -i\hbar\, \varepsilon_{ijk} x_j \frac{\partial f}{\partial x_k}$$

$$= (\text{Const})\, i\hbar\, \varepsilon_{ijk} x_j x_k$$

$$= (\text{Const})\, i\hbar\, (\vec{r}\times\vec{r})_i = 0$$

similarly $[L_j, f(r^2)] = [L_k, f(r^2)] = 0$

$$\Rightarrow [\vec{L}, f(r^2)] = 0$$

Q.E.D

② $\hat{D}\,\psi(\vec{r}) = \psi(\vec{r}-\vec{a})$

a) under infinitesmal translation of $\delta\vec{a}$ we have

$$\psi'(r) = D\,\psi(\vec{r}) = \psi(\vec{r} - \delta\vec{a})$$

now $\psi(\vec{r}-\delta\vec{a}) \simeq \psi(\vec{r}) - \delta\vec{a}\cdot\vec{\nabla}\psi$ ; but $\vec{p} = -i\hbar\vec{\nabla}$

$$\simeq \psi(\vec{r}) - \frac{i}{\hbar}(\delta\vec{a}\cdot\vec{p})\,\psi(r)$$

$$\simeq \left[1 - \frac{i}{\hbar}\,\delta\vec{a}\cdot\vec{p}\right]\psi(\vec{r})$$

$\therefore\ \hat{D}\,\psi(\vec{r}) = \psi(\vec{r}-\delta\vec{a}) = \left[1-\frac{i}{\hbar}\,\delta\vec{a}\cdot\vec{p}\right]\psi(\vec{r})$

$\Rightarrow \boxed{\hat{D} \approx 1 - \frac{i}{\hbar}\,\delta\vec{a}\cdot\vec{p}}$ ; $\hat{p}$ is the generator of translation

ⓑ for finite translation of $\vec{a}$ ; $\vec{a} = N\,\delta\vec{a}$ ; $N \to \infty$

$\therefore\ \hat{D}(\vec{a}) = \lim_{N\to\infty}\{\hat{D}(\delta\vec{a})\}^N$

$$\simeq \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\frac{\vec{a}\cdot\vec{p}}{N}\right]^N \simeq \left[e^{-\frac{i}{\hbar}\frac{\vec{a}\cdot\vec{p}}{N}}\right]^N \simeq e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{p}}$$

ⓒ For H to have a translational symmetry, it is required that $[H, P] = 0$. This requires that both kinetic energy and potential energy commute with H

$\left[\frac{P^2}{2m}, P\right] = 0$ by defult

i.e $[V(x), P] = i\hbar\frac{dV}{dx} = 0$

$$\Rightarrow \frac{dV}{dx} = 0 \quad \Rightarrow \quad V(x) = \text{constant} \equiv V_0$$

(3) $$e^{-i\frac{\alpha}{2}(\sigma\cdot n)} = \sum_{k=0}^{\infty} \frac{(-i\frac{\alpha}{2}\sigma\cdot n)^k}{k!}$$ ; this can be decomposed into an even an odd series

$$= \sum_{k=0}^{\infty} \frac{(-i)^{2k}(\frac{\alpha}{2})^{2k}}{(2k)!}(\sigma\cdot n)^{2k} + \sum_{k=0}^{\infty} \frac{(-i)^{2k+1}(\frac{\alpha}{2})^{2k+1}}{(2k+1)!}(\sigma\cdot n)^{2k+1}$$

Now $(-i)^{2k} = (-1)^{2k}(i)^{2k} = (i^2)^k = (-1)^k$

and $(-i)^{2k+1} = (-1)^{2k+1}(i)^{2k+1} = (-1)(i)^{2k}(i) = (-1)(i^2)^k(i)$

$= -i(-1)^k$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\alpha}{2})^{2k}}{(2k)!}(\sigma\cdot n)^{2k} - i\sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\alpha}{2})^{2k+1}}{(2k+1)!}(\sigma\cdot n)^{2k+1}$$

Now using $(\sigma\cdot n)^n = \begin{cases} I, & \text{even } n \\ \sigma\cdot n, & \text{odd } n \end{cases}$

$\Rightarrow (\sigma\cdot n)^{2k} = I$ and $(\sigma\cdot n)^{2k+1} = \sigma\cdot n$

$$= I\underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\alpha}{2})^{2k}}{(2k)!}}_{\cos\frac{\alpha}{2}} - i(\sigma\cdot n)\underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\alpha}{2})^{2k+1}}{(2k+1)!}}_{\sin\frac{\alpha}{2}}$$

$$= I\cos\left(\frac{\alpha}{2}\right) - i(\sigma\cdot n)\sin\left(\frac{\alpha}{2}\right)$$

when $I$ is the identity matrix, and sometimes it is called $\sigma_0$

(4) $\chi_i = \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\theta/2 \end{pmatrix}$

a) for the eigenvalue $+\frac{\hbar}{2}$ ; $\chi_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

" " " $-\frac{\hbar}{2}$ ; $\chi_f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$P(+\tfrac{\hbar}{2}) = |\langle \chi_f | \chi_i \rangle|^2 = \left| \left\langle (1\ 0) \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \end{pmatrix} \right\rangle \right|^2$$

$$= \left| e^{-i\phi/2} \cos\frac{\theta}{2} \right|^2 = \cos^2\frac{\theta}{2}$$

$$P(-\tfrac{\hbar}{2}) = |\langle \chi_f | \chi_i \rangle|^2 = \left| \left\langle (0\ 1) \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \end{pmatrix} \right\rangle \right|^2$$

$$= \left| e^{i\phi/2} \sin\frac{\theta}{2} \right|^2 = \sin^2\frac{\theta}{2}$$

Notice that $P(+\hbar/2) + P(-\hbar/2) = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$

(b) $\langle \chi_i | S_z | \chi_i \rangle = \left( e^{i\phi/2} \cos\frac{\theta}{2} \quad e^{-i\phi/2} \sin\frac{\theta}{2} \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\theta/2 \end{pmatrix}$

$$= \frac{\hbar}{2} \left( \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \right) = \frac{\hbar}{2} \cos\theta$$

smae result would be obtained if we use $\langle \chi_f | S_z | \chi_f \rangle$

(c) if $\theta = 0 \Rightarrow P(+\hbar/2) = 1$ ; $P(-\hbar/2) = 0$

the beam is polarized in the same direction of the polarizer

$\langle S_z \rangle = \frac{\hbar}{2}$ as expected.

(5) $H = \frac{\hbar^2 l_z^2}{2I}$ ; $l_z = -i\frac{\partial}{\partial\phi}$

need to solve $H\psi = E\psi \Rightarrow -\frac{\hbar^2}{2I}\frac{d^2\psi}{d\phi^2} = E\psi$

$\Rightarrow \frac{d^2\psi}{d\phi^2} + m^2\psi = 0$ ; $m^2 = \frac{2IE}{\hbar^2}$

$\psi_m(\phi) = Ae^{im\phi}$ ; normalize $\int_0^{2\pi} |\psi|^2 d\phi = 1$

$A = \frac{1}{\sqrt{2\pi}}$

$\Rightarrow \psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$

but $\psi$ is periodic and single-valued

$\psi(\phi) = \psi(\phi + 2\pi)$

$\Rightarrow e^{im\phi} = e^{im(\phi+2\pi)} = e^{im\phi} e^{im2\pi}$

$\Rightarrow e^{im2\pi} = 1 \Rightarrow \cos(2\pi m) = 1$

$\Rightarrow 2\pi m = 0, \pm 2\pi, \pm 4\pi, \ldots$

$\Rightarrow m = 0, \pm 1, \pm 2, \ldots$

we see that each level (except $m=0$) is twofold degenerate

⑥ show that $(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \sigma_0 (\vec{a}.\vec{b}) + i\sigma.(\vec{a}\times\vec{b})$

when $\sigma_0$ is the identity matrix

starting from the L.H.S, we have

$$(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \left(\sum_i \sigma_i a_i\right)\left(\sum_j \sigma_j b_j\right)$$

$$= \sum_{i,j} \sigma_i \sigma_j a_i b_j$$

Now using $\sigma_i \sigma_j = \sigma_0 \delta_{ij} + i \sum_k \varepsilon_{ijk} \sigma_k$, we have

$$(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \sum_{i,j} \left(\sigma_0 \delta_{ij} + i \sum_k \varepsilon_{ijk} \sigma_k\right) a_i b_j$$

$$= \sum_{i,j} \delta_{ij} \sigma_0 a_i b_j + i \sum_k \sigma_k \sum_{i,j} \varepsilon_{ijk} a_i b_j$$

$$= \sigma_0 \sum_i a_i b_i + i \sum_k \sigma_k (\vec{a}\times\vec{b})_k$$

$$= \sigma_0 (\vec{a}.\vec{b}) + i\sigma.(\vec{a}\times\vec{b})$$

Q.E.D

Notice that if $\vec{a} = \vec{b} = \vec{n}$ unit vector

$\Rightarrow (\vec{\sigma}.\vec{n})^2 = \sigma_0 = I$ identity