Graduate QM
HW # 1 - solution
Dr. Gassem Alzoubi

① show that $[H, L_z] = 0$; where $H = \frac{P^2}{2m} + V(r)$ → central

$$\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

and $L_z = xP_y - yP_x$

$$[H, L_z] = \left[\frac{P^2}{2m} + V(r), xP_y - yP_x\right] = \left[\frac{P^2}{2m}, xP_y - yP_x\right] + [V(r), xP_y - yP_x]$$

$$= \frac{1}{2m}[P^2, xP_y] - \frac{1}{2m}[P^2, yP_x] + \underbrace{\cancelto{0}{[V(r), xP_y]} - \cancelto{0}{[V(r), yP_x]}}$$

as $[V(r), x_j] = 0$; $j = 1, 2, 3$

and $[V(r), P_j] = 0$; $j = 1, 2, 3$

$$= \frac{1}{2m}\left([P_x^2, xP_y] + [P_y^2, xP_y] + [P_z^2, xP_y]\right)$$

$$- \frac{1}{2m}\left([P_x^2, yP_x] + [P_y^2, yP_x] + [P_z^2, yP_x]\right)$$

$$= \frac{1}{2m}\left(x\cancelto{0}{[P_x^2, P_y]} + [P_x^2, x]P_y + x\cancelto{0}{[P_y^2, P_y]} + \cancelto{0}{[P_y^2, x]}P_y + x\cancelto{0}{[P_z^2, P_y]} + \cancelto{0}{[P_z^2, x]}P_y\right)$$

$$- \frac{1}{2m}\left(y\cancelto{0}{[P_x^2, P_x]} + \cancelto{0}{[P_x^2, y]}P_x + y\cancelto{0}{[P_y^2, P_x]} + [P_y^2, y]P_x + y\cancelto{0}{[P_z^2, P_x]} + \cancelto{0}{[P_z^2, y]}P_x\right)$$

$$= \frac{1}{2m}\begin{bmatrix} P_x[P_x, x]P_y + [P_x, x]P_xP_y \\ - P_y[P_y, y]P_x - [P_y, y]P_yP_x \end{bmatrix}$$

$$= \frac{1}{2m}\left(-i\hbar P_xP_y - i\hbar P_xP_y + i\hbar P_yP_x + i\hbar P_yP_x\right);$$

$= 0$, as $P_xP_y = P_yP_x$

similarly $[H, L_y] = [H, L_x] = 0$

NOW $\vec{L} = L_x^2 + L_y^2 + L_z^2$

$$[H, L^2] = [H, L_x^2 + L_y^2 + L_z^2]$$

$$= [H, L_x^2] + [H, L_y^2] + [H, L_z^2]$$

$$= L_x \cancelto{0}{[H, L_x]} + \cancelto{0}{[H, L_x]} L_x$$

$$+ L_y \cancelto{0}{[H, L_y]} + \cancelto{0}{[H, L_y]} L_y$$

$$+ L_z \cancelto{0}{[H, L_z]} + \cancelto{0}{[H, L_z]} L_z$$

$$= 0$$

Q.E.D

(2) $\psi(r,\theta,\phi) = -B(x+iy)\,e^{-r/2a_0}$; a_0 is Bhor radius

(a)

in spherical $= -B(r\sin\theta\cos\phi + i r\sin\theta\sin\phi)\,e^{-r/2a_0}$

$= -Br(\sin\theta\cos\phi + i\sin\theta\sin\phi)\,e^{-r/2a_0}$

Now using $Y_{1,1} = -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{i\phi} = -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta(\cos\phi + i\sin\phi)$

$= -\left(\frac{3}{8\pi}\right)^{1/2}(\sin\theta\cos\phi + i\sin\theta\sin\phi)$

$\Rightarrow \sin\theta\cos\phi + i\sin\theta\sin\phi = -Y_{11}\left(\frac{8\pi}{3}\right)^{1/2}$

$\Rightarrow \psi = Br\left(\frac{8\pi}{3}\right)^{1/2} Y_{11}\, e^{-r/2a_0}$

Now using $R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2}\frac{1}{a_0}\, r e^{-r/2a_0}$

$\Rightarrow r e^{-r/2a_0} = R_{21}\sqrt{3}\, a_0 (2a_0)^{3/2}$

$\Rightarrow \psi(r,\theta,\phi) = B\frac{\sqrt{8\pi}}{\cancel{\sqrt{3}}} Y_{11} R_{21} \cancel{\sqrt{3}}\, a_0 (2a_0)^{3/2}$

$= B\sqrt{8\pi}\, a_0^{5/2}(2)^{3/2} Y_{11} R_{21} = 8B\sqrt{\pi}\, a_0^{5/2} Y_{11} R_{21}$

$\Rightarrow n=2,\ l=1,\ m=1$

(b) $\int |\psi|^2 d^3r = 1 \Rightarrow 64B^2\pi a_0^5 \underbrace{\int_0^\infty r^2 |R_{21}|^2 dr}_{1} \underbrace{\int_\Omega d\Omega\, |Y_{11}|^2}_{1} = 1$

$\Rightarrow B^2 = \frac{1}{64\pi a_0^5}$

$B = \frac{1}{8\sqrt{\pi}\, a_0^{5/2}} \Rightarrow \psi = 8\frac{1\sqrt{\pi}}{8\sqrt{\pi}\, a_0^{5/2}} a_0^{5/2} Y_{11} R_{21} = R_{21}(r) Y_{11}(\theta,\phi)$

ⓒ $\therefore\ \psi(r,\theta,\phi) = R_{21}(r)\, Y_{11}(\theta,\phi)$

$$\Rightarrow \langle r\rangle = \int_0^\infty r^3 |R_{21}|^2 dr \ ; \ \text{where } R_{21}(r) = \frac{1}{\sqrt{24}} a_0^{-3/2} \frac{r}{a_0} e^{-r/2a_0}$$

$$= \frac{1}{24}\frac{1}{a_0^5}\int_0^\infty r^5 e^{-r/a_0} dr = \frac{1}{24}\frac{1}{a_0^5}\frac{5!}{(1/a_0)^6}$$

$$= \frac{5!}{24}\frac{a_0^6}{a_0^5} = 5a_0$$

ⓓ $P_{21}(r) = r^2 |R_{21}(r)|^2 = r^2 \frac{1}{24}\frac{1}{a_0^5} r^2 e^{-r/a_0}$

$$\frac{dP_{21}}{dr} = 0 = \frac{1}{24} a_0^5 \left[r^4\left(-\frac{1}{a_0}\right) e^{-r/a_0} + 4r^3 e^{-r/a_0} \right]$$

$$\Rightarrow \quad 4r^3 \cancel{e^{-r/a_0}} = \frac{r^4}{a_0} \cancel{e^{-r/a_0}}$$

$$\Rightarrow \quad 4 = \frac{r}{a_0}$$

$$\Rightarrow \quad \boxed{r = 4a_0}$$

(3) $\Psi(r,\theta,\phi) = 2\psi_{100} + \psi_{210}$; $\Psi^*(r,\theta,\phi) = 2\psi_{100}^* + \psi_{210}^*$

a) Normalize $\Psi(r,\theta,\phi) = A[2\psi_{100} + \psi_{210}]$

$$\int_{-\infty}^{\infty} |\Psi(r,\theta,\phi)|^2 d^3r = 1 = \int d^3r\,(2\psi_{100}^* + \psi_{210}^*)(2\psi_{100} + \psi_{210})$$

$$\Rightarrow 4A^2 \underbrace{\int_{-\infty}^{\infty} |\psi_{100}|^2 d^3r}_{1} + A^2 \underbrace{\int_{-\infty}^{\infty} |\psi_{210}|^2 d^3r}_{1} = 1 \quad \Rightarrow 5A^2 = 1$$

$$\Rightarrow A = \frac{1}{\sqrt{5}}$$

b) $P(n=3) = 0$, electron exists only in either $n=1$ or $n=2$

c) $n=1 \Rightarrow E_1 = -\frac{13.6}{1^2} = -13.6\,eV$; $E_2(n=2) = \frac{-13.6}{4} = -3.4\,eV$

$\Downarrow$ $\qquad\qquad\qquad\qquad\qquad \downarrow$

$l=0$ $\qquad\qquad\qquad\qquad\qquad l=0,1$

$$\Rightarrow L = \sqrt{l(l+1)}\,\hbar = \begin{cases} 0, & l=0 \\ \sqrt{2}\,\hbar, & l=1 \end{cases}$$

Possible values of L_z is $m_l\hbar$; where $m_l = 0$ only as show in the wave function

$\therefore L_z = 0\,\hbar$

$2\psi_{100} + \psi_{210} \rightarrow m=0$

d) $$P(E_1) = P(l=0) = \left|\langle\psi_{100}|\tfrac{2}{5}\psi_{100} + \tfrac{1}{\sqrt{5}}\psi_{210}\rangle\right|^2 = \left|\tfrac{2}{\sqrt{5}}\right| = \tfrac{4}{5}$$

$$P(E_2) = P(L=\sqrt{2}\hbar) = \left|\langle\psi_{210}|\tfrac{2}{\sqrt{5}}\psi_{100} + \tfrac{1}{\sqrt{5}}\psi_{210}\rangle\right|^2 = \left|\tfrac{1}{\sqrt{5}}\right|^2 = \tfrac{1}{5}$$

Now $P(L_z=0) = 1$ as $L_z = 0$ is the only possible m value

(4) $V(r) = -g\,\delta(r-a)$; $g>0$

the radial eq^n reads

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right\} R(r) = E\,R(r)$$

let $R(r) = \dfrac{\psi(r)}{r}$

$$\Rightarrow -\frac{\hbar^2}{2mr^2}\underbrace{\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{\psi}{r}\right)\right)} + \left[V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right]\frac{\psi}{r} = E\,\frac{\psi}{r}$$

$$= \frac{d}{dr}\left(r^2\left(\frac{1}{r}\frac{d\psi}{dr} - \frac{1}{r^2}\psi\right)\right) = \frac{d}{dr}\left(r\frac{d\psi}{dr} - \psi\right)$$

$$= r\frac{d^2\psi}{dr^2} + \cancel{\frac{d\psi}{dr}} - \cancel{\frac{d\psi}{dr}} = r\frac{d^2\psi}{dr^2}$$

$$\frac{-\hbar^2}{2mr^2}\,r\frac{d^2\psi}{dr^2} + \left[V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right]\frac{\psi}{r} = E\frac{\psi}{r}$$

multiply by r $\Rightarrow -\dfrac{\hbar^2}{2m}\psi'' + \left[V(r) + \dfrac{\hbar^2 l(l+1)}{2mr^2}\right]\psi = E\psi$

Now for Ground state $l=0$ and $E<0$, where $V(r)=0$ for $r\neq a$

so for $r\neq a \Rightarrow$ the equation reads

$-\dfrac{\hbar^2}{2m}\psi'' - E\psi = 0 \Rightarrow \psi'' + \dfrac{2mE}{\hbar^2}\psi = 0$; where $E = -\dfrac{\hbar^2 k^2}{2m}$

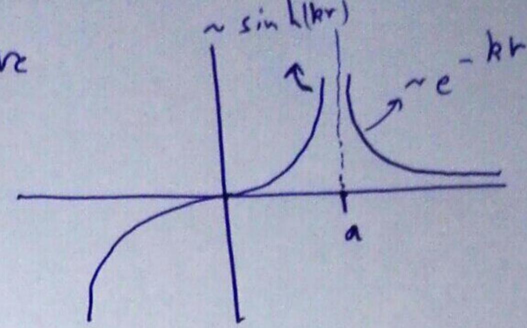
$\Rightarrow \psi'' - k^2\psi = 0$, with the Boundary conditions that

$\psi(r) \to 0$ as $r\to 0$ and as $r\to\infty$

the wave functions that satisfy these B.Cs are

$$\psi(r) = \begin{cases} A\sinh(kr) & ; \quad 0\le r\le a \\ B\,e^{-kr} & ; \quad r>a \end{cases}$$

Now at $r=a$, the wave functions are continuous



$$A \sinh(ka) = Be^{-ka} \quad \text{----- (1)}$$

and the discontinuity of the first derivative at $r=a$ gives

$$\psi_+'(a+\varepsilon) - \psi_+'(a-\varepsilon) = -\frac{2mg}{\hbar^2}\psi_+(a)$$

$$-kBe^{-ka} - kA\cosh(ka) = -\frac{2mg}{\hbar^2}Be^{-ka}$$

$$\Rightarrow B\left(\frac{2mg}{k\hbar^2} - 1\right)e^{-ka} = A\cosh(ka) \quad \text{---- (2)}$$

divide 2 by 1 yields

$$\coth(ka) = \frac{2mg}{k\hbar^2} - 1$$

multiply by ka $\Rightarrow$ $ka\coth(ka) = \frac{2mag}{\hbar^2} - ka$

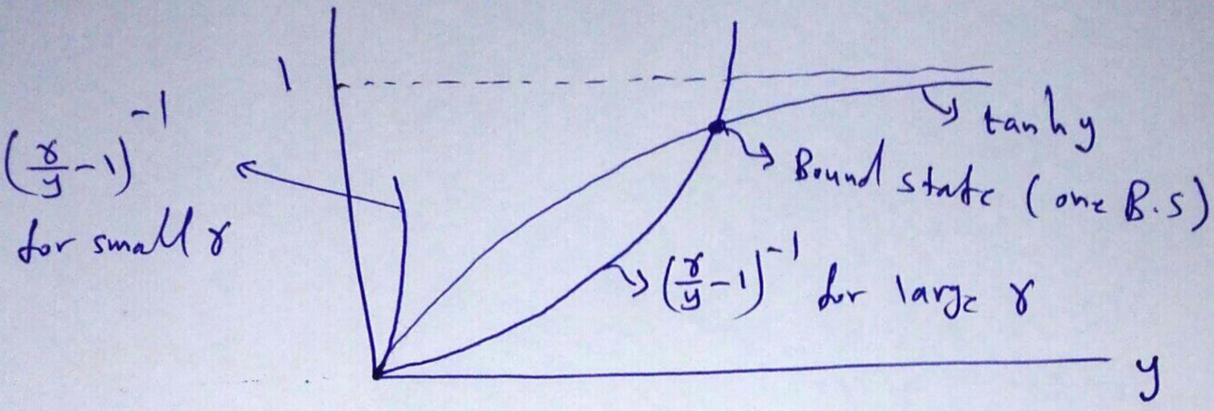
let $y = ka$ $\Rightarrow$ $y\coth y = \frac{2mag}{\hbar^2} - y$;

$$y\coth y = \gamma - y \quad ; \quad \text{where } \gamma = \frac{2mg}{\hbar^2}$$

$$\coth y = \frac{\gamma}{y} - 1$$

$$\tanh y = \left(\frac{\gamma}{y} - 1\right)^{-1}$$

let us solve this eq^n Graphically



the intersection occurs only if the slope of $\left(\frac{\gamma}{y}-1\right)^{-1}$ at $y=0$ is smaller than the slope of tanh y

i.e $\left.\frac{d}{dy}\left(\frac{\gamma}{y}-1\right)^{-1}\right|_{y=0} < \left.\frac{d}{dy}\tanh\right|_{y=0}$

$$\left.\frac{(\gamma-y)(-1)-y(-1)}{(\gamma-y)^2}\right|_{y=0} < \left.1-\tanh^2 y\right|_{y=0}$$

$$\left.\frac{\gamma}{(\gamma-y)^2}\right|_{y=0} < \left.1-\tanh^2 y\right|_{y=0}$$

$$\frac{1}{\gamma} < 1 \quad \text{or} \quad \gamma > 1$$

$$\frac{2mag}{\hbar^2} > 1$$

$$g > \frac{\hbar^2}{2ma}$$

this is the range of g that supports one bound state

(5) in Polar coordinates $x = \rho\cos\phi$; $y = \rho\sin\phi$

and $\rho^2 = x^2 + y^2$

$$\nabla^2 = \frac{1}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} \quad \text{or } \nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

the schrodinger eq^n reads

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\rho)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\psi + V(\rho)\psi = E\psi \quad \text{---- (1)}$$

let $\Psi(\rho,\phi) = R(\rho)\,\Phi(\phi)$; substitute this in (1)

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\Phi\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\rho^2}R\frac{\partial^2\Phi}{\partial\phi^2}\right] + V(\rho)R\Phi = ER\Phi$$

multiply by ρ^2 and divide by $\Psi = R\Phi$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} - \frac{2m}{\hbar^2}V(\rho)\rho^2 + \frac{2m}{\hbar^2}E\rho^2 = 0$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{2m}{\hbar^2}(E - V(\rho))\rho^2 = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \quad \text{---- (2)}$$

the L.H.S in a function of ρ and the R.H.S is a function of ϕ, so they are equal if and only if they both equal the same constant. let us call this constant m^2

$$\Rightarrow -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = m^2 \Rightarrow \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$\Rightarrow \Phi(\phi) = Ae^{im\phi}$; normalize $\int_0^{2\pi}|\Phi|^2 d\phi = 1$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$\therefore \quad \Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$

but Φ is periodic and single-valued

$\Rightarrow \Phi(2\pi+\phi) = \Phi(\phi) \Rightarrow e^{im(2\pi+\phi)} = e^{im\phi}$

$\Rightarrow e^{im2\pi} = 1 \Rightarrow \cos(2\pi m) = 1 \Rightarrow 2\pi m = 0, \pm 2\pi, \pm 4\pi, \ldots$

$m = 0, \pm 1, \pm 2, \pm 3, \ldots$

Now the radial eq^n reads

$$\frac{\rho}{R}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \frac{2m}{\hbar^2}(E - V(\rho))\rho^2 = m^2$$

$$\frac{\rho}{R}\left[\frac{dR}{d\rho} + \rho \frac{d^2R}{d\rho^2}\right] + \frac{2m}{\hbar^2}(E - V(\rho))\rho^2 = m^2$$

multiply by R/ρ^2

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \frac{2m}{\hbar^2}(E - V(\rho))R = \frac{m^2 R}{\rho^2}$$

$$\boxed{\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{m^2 R}{\rho^2} + \frac{2m}{\hbar^2}(E - V(\rho))R = 0}$$