Coupling of orbital and spin angular momenta:

Let us consider the problem of adding the orbital angular momentum l to a spin of a particle. We know that L^2, S^2, J^2, J_z form a CSCO. Let us take the special case of $s = 1/2$.

$\Rightarrow \; j_1 = l \Rightarrow m_1 = m_l \; ; \; (2j_1+1) = (2l+1)$ states

$j_2 = s = 1/2 \Rightarrow m_2 = m_s = \pm 1/2 \; ; \; (2j_2+1) = 2$ states

the combined space has a dimension of $2(2l+1)$ and the allowed j values are

$$|j_1 - j_2| \leq j \leq |j_1 + j_2|$$

$$l - 1/2 \leq j \leq l + 1/2$$

two spin states $\uparrow$ and $\downarrow$

so if $l = 0$, we have only two possible values $j = l \pm 1/2$

Now if $l > 0$; j can take only two values

- for example if $l = 1 \Rightarrow 1/2 \leq j \leq 3/2 \Rightarrow 1/2, 3/2$ two states

if $l = 2 \Rightarrow 3/2 \leq j \leq 5/2 \Rightarrow 3/2, 5/2$ two states

and so on

so $j = l + 1/2$ or $l - 1/2$

$\searrow (2j+1) = (2l - 1 + 1) = 2l$ states

$(2j+1) = (2l+1+1)$
$= 2l + 2$
$= 2(l+1)$ states

total $= 2l + 2 + 2l = 2(2l+1)$ as expected

Now let us study the subspaces $j = l \pm 1/2$ in details.

a) the subspace $j = l + 1/2$

$$|j,m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j} \langle j_1 m_1\, j_2 m_2 | j m\rangle \, | j_1 m_1\, j_2 m_2\rangle$$

now with $j_1 = l$; $j_2 = 1/2$; $j = l + 1/2$

$$|l+1/2, m\rangle = \sum_{m_l=-l}^{+l} \sum_{m_s=-1/2}^{+1/2} \langle l, m_l, 1/2, m_s | l+1/2, m\rangle \, |l, m_l, 1/2, m_s\rangle$$

$$= \sum_{m_l=-l}^{+l} \langle l, m_l, 1/2, -1/2 | l+1/2, m\rangle \, |l, m_l, 1/2, -1/2\rangle$$

$$+ \sum_{m_l=-l}^{+l} \langle l, m_l, 1/2, +1/2 | l+1/2, m\rangle \, |l, m_l, 1/2, +1/2\rangle$$

now let $m = m_1 + m_2$;
$m = m_l + m_s$ $\Rightarrow m_l = m - m_s = m - (\mp 1/2)$

$$\Rightarrow |l+1/2, m\rangle = \underbrace{\langle l, m+1/2, 1/2, -1/2 | l+1/2, m\rangle}_{C^{l+1/2,m}_{l,m+1/2,1/2,-1/2}} \, |l, m+1/2, 1/2, -1/2\rangle$$

$$+ \underbrace{\langle l, m-1/2, 1/2, +1/2 | l+1/2, m\rangle}_{C^{l+1/2,m}_{l,m-1/2,1/2,1/2}} \, |l, m-1/2, 1/2, +1/2\rangle$$

$$C^{l+1/2,m}_{l,m+1/2,1/2,-1/2} = \sqrt{\frac{l-m+1/2}{2l+1}} \qquad C^{l+1/2,m}_{l,m-1/2,1/2,1/2} = \sqrt{\frac{l+m+1/2}{2l+1}}$$

can be found in many Q.M references

where $m = -l-1/2, -l+1/2, -l+3/2, \ldots\ l-3/2, l-1/2, l+1/2$

$$\Rightarrow |l+1/2, m\rangle = \sqrt{\frac{l-m+1/2}{2l+1}} \, |l, m+1/2, 1/2, -1/2\rangle + \sqrt{\frac{l+m+1/2}{2l+1}} \, |l, m-1/2, 1/2, +1/2\rangle$$

if the electron moves in a central potential (like H atom)
its complete wavefunction consists of a space part $(n, l, m \pm 1/2)$
$R_{nl}(r)\, Y_{l, m\pm 1/2}$ and a spin part $|1/2, \pm 1/2\rangle$

$$\Rightarrow |l+1/2, m\rangle = \sqrt{\frac{l - m + 1/2}{2l+1}}\; \underbrace{|l, m+1/2\rangle}_{Y_{l,m+1/2}}\; \underbrace{|1/2, -1/2\rangle}_{\chi_{+}}$$

$$+ \sqrt{\frac{l+m+1/2}{2l+1}}\; \underbrace{|l, m-1/2\rangle}_{Y_{l,m-1/2}}\; \underbrace{|1/2, +1/2\rangle}_{\chi_+}$$

$$\Rightarrow \psi_{n,l,j=l+1/2,m} = R_{nl}(r) \left[\sqrt{\frac{l-m+1/2}{2l+1}}\; Y_{l,m+1/2}\; \underbrace{|1/2, -1/2\rangle}_{\chi_+} \right.$$

$$\left. + \sqrt{\frac{l+m+1/2}{2l+1}}\; Y_{l,m-1/2}\; \underbrace{|\tfrac{1}{2}, +1/2\rangle}_{\chi_-} \right]$$

b) the subspace $j = l - 1/2$ one can find
following the same procedure,

$$|l-1/2, m\rangle = \sqrt{\frac{l+m+1/2}{2l+1}}\; |l, m+1/2\rangle\, |1/2, -1/2\rangle$$

$$+ \sqrt{\frac{l-m+1/2}{2l+1}}\; |l, m-1/2\rangle\, |1/2, +1/2\rangle$$

and

$$\psi_{n,l,j=l-1/2,m} = R_{nl}(r) \left[\sqrt{\frac{l+m+1/2}{2l+1}}\; Y_{l,m+1/2}\; \chi_+ \right.$$

$$\left. + \sqrt{\frac{l-m+1/2}{2l+1}}\; Y_{l,m-1/2}\; \chi_- \right\}$$

Addition of more than Two angular Momenta

consider $J_1, J_2, J_3 \Rightarrow J = J_1 + J_2 + J_3$

Add J_1 and J_2 first $\Rightarrow J_{12}$

then add J_{12} to J_3

$\therefore \vec{J}_1 + \vec{J}_2 \Rightarrow \quad |j_1 - j_2| \leq j_{12} \leq |j_1 + j_2|$; $m_{12} = m_1 + m_2$

then $\vec{J}_{12} + J_3 \Rightarrow \quad |j_{12} - j_3| \leq j \leq |j_{12} + j_3|$; $m = m_{12} + m_3$

this procedure can be Generalized to N momenta

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \cdots + \vec{J}_N$$

Example : Adding 3 spin $1/2$ $\quad 1/2 + 1/2 + 1/2$

$j_1 = 1/2, \quad j_2 = 1/2, \quad j_3 = 1/2$

$j_{12} = j_1 + j_2 \Rightarrow \quad |j_1 - j_2| \leq j_{12} \leq |j_1 + j_2| \Rightarrow 0 \leq j_{12} \leq 1$

$\Rightarrow \quad j_{12} = 0, 1$

Now $j_{12} + j_3 \Rightarrow \quad |j_{12} - j_3| \leq j \leq |j_{12} + j_3|$

for $j_{12} = 0 \Rightarrow \quad 1/2 \leq j \leq 1/2 \Rightarrow j = 1/2$

for $j_{12} = 1 \Rightarrow \quad 1/2 \leq j \leq 3/2 \Rightarrow j = 1/2, 3/2$

$\therefore \quad j = 1/2, 3/2$