Addition of angular momenta

consider two angular momenta $\vec{J_1}$ and $\vec{J_2}$ which belong to two different subspaces 1 and 2. J_1 and J_2 may refer to two distinct particles of two different properties of the same particle (like $\vec{L}$ and $\vec{S}$ of electron in H atom).

since J_1 and J_2 belong to two distinct spaces $\Rightarrow [\vec{J_1}, \vec{J_2}] = 0$

- for J_1
$$\left.\begin{aligned} J_1^2 |j_1, m_1\rangle &= j_1(j_1+1)\hbar^2 |j_1, m_1\rangle \\ J_{1z} |j_1, m_1\rangle &= \hbar m_1 |j_1, m_1\rangle \end{aligned}\right\} \text{dimension } (2j_1+1)$$

- for J_2
$$\left.\begin{aligned} J_2^2 |j_2, m_2\rangle &= j_2(j_2+1)\hbar^2 |j_2, m_2\rangle \\ J_{2z} |j_2, m_2\rangle &= \hbar m_2 |j_2, m_2\rangle \end{aligned}\right\} \text{dimension } (2j_2+1)$$

the bases $|j_1, m_1\rangle$ are complete and orthogonal

$$\sum_{m_1} |j_1 m_1\rangle\langle j_1 m_1| = 1 \quad \text{and} \quad \langle j_1' m_1' | j_1 m_1\rangle = \delta_{j_1 j_1'} \delta_{m_1 m_1'}$$

similarly for the bases $|j_2, m_2\rangle$

$$\sum_{m_2} |j_2 m_2\rangle\langle j_2 m_2| = 1 \quad \text{and} \quad \langle j_2' m_2' | j_2 m_2\rangle = \delta_{j_2 j_2'} \delta_{m_2 m_2'}$$

- Notice that the four operators $J_1^2, J_{1z}, J_2^2, J_{2z}$ form a complete set of commuting operators (C.S.C.O); they can be diagonalized by the same eigenstates. let us denote them by ~~$|j_1, j_2, m$~~ (shortly $|j_1 m_1; j_2 m_2\rangle$)

$$|j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle |j_2, m_2\rangle$$

$$J_1^2 |j_1, m_1; j_2, m_2\rangle = j_1(j_1+1)\hbar^2 |j_1 m_1; j_2 m_2\rangle$$
$$J_{1z} |j_1 m_1 j_2 m_2\rangle = \hbar m_1 |j_1 m_1 j_2 m_2\rangle$$
$$J_2^2 |j_1 m_1 j_2 m_2\rangle = j_2(j_2+1)\hbar^2 |j_1 m_1 j_2 m_2\rangle$$

and $J_{2z}|j_1m_1 j_2m_2\rangle = \hbar m_2 |j_1m_1 j_2m_2\rangle$

the kets $|j_1m_1 j_2m_2\rangle$ form a complete and orthogonal bases

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1m_1 j_2m_2\rangle\langle j_1m_1 j_2m_2| = 1$$

and $\langle j_1'm_1' j_2'm_2' | j_1m_1 j_2m_2\rangle = \delta_{j_1j_1'}\delta_{m_1m_1'}\delta_{j_2j_2'}\delta_{m_2m_2'}$

The concern is now how to find the eigenvalues and the eigenstates of the new set of J^2, J_z, J_1, J_2 in terms of the old eigenvalues and eigenstates of $J_1^2, J_2^2, J_{1z}, J_{2z}$. the new set of J^2, J_z, J_1, J_2 form a complete set i.e they can be diagonalized by the same eigenstates. let us denote them by $|j_1, j_2; jm\rangle$. since j_1 and j_2 are usually fixed, we will use a short hand notation ($|jm\rangle$)

$\therefore$ $J_1^2|jm\rangle = j_1(j_1+1)\hbar^2|jm\rangle$

$J_2^2|jm\rangle = j_2(j_2+1)\hbar^2|jm\rangle$

$J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$

$J_z|jm\rangle = \hbar m|jm\rangle$

the dimension of the combined ~~system~~ space is $(2j_1+1)(2j_2+1)$

so for every j, the # m has $(2j+1)$ allowed values

$$\underset{m_{min}}{m = -j}, -j+1, \cdots, j-1, \underset{m_{max}}{j} \qquad -j \leq m \leq j$$

of course $|j_1-j_2| \leq j \leq j_1+j_2$

notes that the new states $|jm\rangle$ are also complete and orthonormal.

now how can we express the new bases $|j, m\rangle$ in terms of the composite bases ~~$|j_1, j_2\, m_1, m_2\rangle$~~ $|j_1 m_1\, j_2 m_2\rangle$

$$|j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1\, j_2 m_2 | j\, m\rangle$$

$$= \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | j m\rangle \; |j_1 m_1\, j_2 m_2\rangle$$

$$|j, m\rangle = \sum_{m_1 m_2} C^{jm}_{j_1 m_1\, j_2 m_2} \; |j_1 m_1\, j_2 m_2\rangle$$

Clebsch - Gordan coefficients (always real)

or can be writtens $|j, m\rangle = \sum_{m_1 m_2} C^{jm}_{j_1 m_1 j_2 m_2} \; |j_1 m_1\rangle |j_2 m_2\rangle$

some properties of $C^{jm}_{j_1 m_1 j_2 m_2}$

1) if $m \neq m_1 + m_2 \Rightarrow \langle j_1 m_1 j_2 m_2 | j m\rangle = 0$
   if $m = m_1 + m_2 \Rightarrow \langle j_1 m_1 j_2 m_2 | j m\rangle \neq 0$ finite } Selection Rule

2) if $m_1 = j_1$; $m_2 = j_2$; $j = j_1 + j_2$; $m = j_1 + j_2$ (uper level)

$$\langle j_1 m_1\, j_2 m_2 | j m\rangle = \langle j_1 j_1\, j_2 j_2 | j m\rangle = 1$$

3) if $m_1 = -j_1$; $m_2 = -j_2$; $j = j_1 + j_2$; $m = -(j_1 + j_2)$ bottom level

$$\langle j_1 m_1\, j_2 m_2 | j m\rangle = \langle j_1 j_1\, j_2 j_2 | j m\rangle = 1$$

Example: Addition of $j_1 = 1$ and $j_2 = 1$

the total dimension of the combined space $= (2j_1+1)(2j_2+1)$

$= 3 \times 3 = 9$ states

$j_1 + j_2 = 2$; $|j_1 - j_2| = 0$

$\Rightarrow \quad 0 \leq j \leq 2$

$\Rightarrow \quad j = 2, 1, 0 \longrightarrow$ one state $j = 0$, $m = 0$

5 states $j = 2$, $m = 2, 1, 0, -1, -2$

three states $j = 1$, $m = 1, 0, -1$

a) the subspace $j = 2$

upper state is $|2,2\rangle = |1,1,1,1\rangle = |j_1 m_1 j_2 m_2\rangle$

only one way to obtain this state

$\Rightarrow \quad C^{22}_{1111} = 1$

Apply J_- to this state to obtain $|2,1\rangle$

$$J_- |2,2\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar\,|2,1\rangle = 2\hbar\,|2,1\rangle$$

$$\Rightarrow |2,1\rangle = \frac{1}{2\hbar} J_- |2,2\rangle = \frac{1}{2\hbar}(J_{1-} + J_{2-})|1,1,1,1\rangle$$

$$= \frac{1}{2\hbar}\left[J_{1-}|1,1,1,1\rangle + J_{2-}|1,1,1,1\rangle\right]$$

$j_1 = 1$, $j_2 = 1$

$$= \frac{1}{2\hbar}\Big[\sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\hbar\,|1,0,1,1\rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\hbar\,|1,1,1,0\rangle\Big]$$

$$= \frac{1}{2\hbar}\left[\sqrt{2}\hbar\,|1,0,1,1\rangle + \sqrt{2}\hbar\,|1,1,1,0\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[|1,0,1,1\rangle + |1,1,1,0\rangle\right]$$

$$\Rightarrow C^{21}_{1011} = C^{21}_{1110} = \frac{1}{\sqrt{2}}$$

apply J_- again to $|2,1\rangle$

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left\{|1,1,1,-1\rangle + 2|1,0,1,0\rangle + |1,-1,1,1\rangle\right\}$$

$$\Rightarrow C^{20}_{111-1} = \frac{1}{\sqrt{6}} = C^{20}_{1-111} \quad \text{and} \quad C^{20}_{1010} = \frac{2}{\sqrt{6}}$$

similarly $|2,-1\rangle = \frac{1}{\sqrt{2}}\left[|1,0,1,-1\rangle + |1,-1,1,0\rangle\right\}$

$$C^{2-1}_{101-1} = C^{2-1}_{1-110} = \frac{1}{\sqrt{2}}$$

and finally $|2,-2\rangle = |1,-1,1,-1\rangle$; $C^{2-2}_{1-11-1} = 1$

b) the subspace of $j=1$

upper state $|1,1\rangle$ can be obtained by two ways

$|1,1,1,0\rangle$ and $|1,0,1,1\rangle$

$\Rightarrow |1,1\rangle = \alpha|1,1,1,0\rangle + \beta|1,0,1,1\rangle$ with $\alpha^2+\beta^2=1$

we need another eq^n to calculate α, β completely

from orthogonality of $|1,1\rangle$ with $|2,1\rangle$

$$\langle 2,1|1,1\rangle = \frac{\beta}{\sqrt{2}} + \frac{\alpha}{\sqrt{2}} = 0 \Rightarrow \alpha+\beta=0 \Rightarrow \beta=-\alpha$$

from $\alpha^2+\beta^2=1 \Rightarrow \alpha^2+\alpha^2=1 \Rightarrow \alpha=\frac{1}{\sqrt{2}}$; $\beta=-\frac{1}{\sqrt{2}}$

$$\Rightarrow |1,1\rangle = \frac{1}{\sqrt{2}}\left[|1,1,1,0\rangle - |1,0,1,1\rangle\right\}$$

$$\Rightarrow C^{11}_{1110} = \frac{1}{\sqrt{2}} ; \quad C^{11}_{1011} = -\frac{1}{\sqrt{2}}$$

Now apply J_-

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left\{|1,1,1,-1\rangle - |1,-1,1,1\rangle\right\}$$

$$\Rightarrow C^{10}_{111-1} = \frac{1}{\sqrt{2}} ; \quad C^{10}_{1-111} = -\frac{1}{\sqrt{2}}$$

and finally $|1,-1\rangle = \frac{1}{\sqrt{2}}\left\{|1,0,1,-1\rangle - |1,-1,1,0\rangle\right\}$

$$\Rightarrow C^{1-1}_{101-1} = \frac{1}{\sqrt{2}} ; \quad C^{1-1}_{1-110} = -\frac{1}{\sqrt{2}}$$

c) the subspace $j=0$, one state $|0,0\rangle$

there are 3 ways to obtain this state

$$|0,0\rangle = a\,|1,1,1,-1\rangle + b\,|1,0,1,0\rangle + c\,|1,-1,1,1\rangle$$

with $a^2+b^2+c^2=1$

This state must be orthogonal to $|2,0\rangle$ and $|1,0\rangle$

thus yielding $a+2b+c=0$ and $a-c=0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\leftarrow a=c$$

$$b+c=0$$

$$\Rightarrow b=-c$$

now $c^2+c^2+c^2=1 \Rightarrow 3c^2=1 \Rightarrow c=\frac{1}{\sqrt{3}} \Rightarrow a=\frac{1}{\sqrt{3}}$; $b=-\frac{1}{\sqrt{3}}$

$$\Rightarrow |0,0\rangle = \frac{1}{\sqrt{3}}\left[|1,1,1,-1\rangle - |1,0,1,0\rangle + |1,-1,1,1\rangle\right]$$

<u>Example</u>: Addition of $s_1=1/2$ with $s_2=1/2$

or $j_1=1/2$ with $j_2=1/2$

$j=1,0$

$\swarrow$ 3 states (triplet) $m=1,0,-1$

$\searrow$ singlet state $m=0$

overall we have 4 states

~~two~~ how this work?

for s_1 there are two possibilities $\uparrow_1, \downarrow_1$

for s_2 " " " " $\uparrow_2, \downarrow_2$

so $1/2 \otimes 1/2$

$\left.\begin{matrix}\uparrow_1\uparrow_2\\ \downarrow_1\downarrow_2\end{matrix}\right]$ they intrinsic symmetry

$\left.\begin{matrix}\uparrow_1\downarrow_2\\ \downarrow_1\uparrow_2\end{matrix}\right]$ don't have certain symmetry the last two states can be combined in a way to make one of them symmetric under spin exchange and the other is anti symmetric

$(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2)$ symmetric

$(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$ anti symmetric

Normalize $\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2)$ and $\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$

so for symmetric states, we have

$$\left.\begin{array}{ll} \uparrow_1\uparrow_2 & S_z = 1 \\ \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) & S_z = 0 \\ \downarrow_1\downarrow_2 & S_z = -1 \end{array}\right\} \begin{array}{l} \text{triplet} \\ s = 1 \end{array}$$

$$\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2) \quad S_z = 0 \Big\} \begin{array}{l} \text{singlet state} \\ s = 0 \end{array}$$

Now some Algebra of two spin 1/2 system

$$\vec{S} = \vec{S}_1 + \vec{S}_2 \Rightarrow \vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$\hbar^2 s(s+1) = s_1(s_1+1)\hbar^2 + s_2(s_2+1)\hbar^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$\hbar^2 s(s+1) = \frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$s(s+1) = \frac{3}{2} + \frac{2}{\hbar^2}\vec{S}_1\cdot\vec{S}_2$$

$$= \frac{3}{2} + \frac{2}{\hbar^2}\vec{S}_1\cdot\vec{S}_2$$

$$\Rightarrow \frac{\vec{S}_1\cdot\vec{S}_2}{\hbar^2} = \frac{s(s+1) - 3/2}{2} \quad ; \text{ using } \vec{S}_1 = \frac{\hbar}{2}\vec{\sigma}_1 \,;\ \vec{S}_2 = \frac{\hbar}{2}\vec{\sigma}_2$$

$$\Rightarrow \vec{\sigma}_1\cdot\vec{\sigma}_2 = 2s(s+1) - 3 = \begin{cases} -3\,, & s = 0 \text{ singlet} \\ +1\,, & s = 1 \text{ triplet} \end{cases}$$

Again, the four states can be written as

$$\chi_+^1\chi_+^2,\ \frac{1}{\sqrt{2}}(\chi_+^1\chi_-^2+\chi_-^1\chi_+^2),\ \chi_-^1\chi_-^2,\ \frac{1}{\sqrt{2}}(\chi_+^1\chi_-^2-\chi_-^1\chi_+^2)$$

so

$$S_z\,\chi_+^1\chi_+^2=(S_{1z}+S_{2z})\chi_+^1\chi_+^2=\frac{\hbar}{2}\chi_+^1\chi_+^2+\frac{\hbar}{2}\chi_+^1\chi_+^2=\hbar\,\underbrace{\chi_+^1\chi_+^2}_{m=+1}$$

$$S_z\,\chi_-^1\chi_-^2=(S_{1z}+S_{2z})\chi_-^1\chi_-^2=\underbrace{-\hbar\,\chi_-^1\chi_-^2}_{m=-1}$$

$$S_z\frac{1}{\sqrt{2}}(\chi_+^1\chi_-^2+\chi_-^1\chi_+^2)=\frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}\chi_+^1\chi_-^2-\frac{\hbar}{2}\chi_-^1\chi_+^2-\frac{\hbar}{2}\chi_+^1\chi_-^2+\frac{\hbar}{2}\chi_-^1\chi_+^2\right)$$

$$=0 \longrightarrow m=0$$

finally $S_z\frac{1}{\sqrt{2}}(\chi_+^1\chi_-^2-\chi_-^1\chi_+^2)=0 \longrightarrow m=0\quad (s=0)$

- notice that we can move between the states up and down using the raising and lowering operators $S_\pm$

$S_-=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$; $\chi_+=\begin{pmatrix}1\\0\end{pmatrix}$; $\chi_-=\begin{pmatrix}0\\1\end{pmatrix}$

$S_-\chi_+=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\hbar\begin{pmatrix}0\\1\end{pmatrix}=\hbar\chi_-\quad\therefore S_-\chi_+=\hbar\chi_-$

and $S_-\chi_-=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\hbar\begin{pmatrix}0\\0\end{pmatrix}=0\quad\therefore S_-\chi_-=0$

~~also we can~~ $|s,m\rangle=\begin{cases}\text{triplet} & |1,1\rangle,|1,0\rangle,|1,-1\rangle\\ \text{singlet} & |0,0\rangle\end{cases}$

start from the upper level $|1,1\rangle$ and apply S_- ;

where $S_-|s,m\rangle=\hbar\sqrt{s(s+1)-m(m-1)}\,|s,m-1\rangle$

$\therefore \; S_- |1,1\rangle = \sqrt{2}\hbar\, |1,0\rangle \;\Rightarrow\; |1,0\rangle = \frac{1}{\sqrt{2}\hbar} S_- |1,1\rangle$

Now need to find $S_- |1,1\rangle$

$$S_- |1,1\rangle = (S_{1-} + S_{2-})\, \chi_+^1 \chi_+^2 = (S_{1-}\chi_+^1)\chi_+^2 + \chi_+^1 (S_{2-}\chi_+^2)$$

$|1,1\rangle \downarrow \chi_+^1 \chi_+^2$

$$= \hbar\, \chi_-^1 \chi_+^2 + \hbar\, \chi_+^1 \chi_-^2$$

$$= \frac{\sqrt{2}}{\sqrt{2}} \hbar\, (\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2)$$

$$= \sqrt{2}\hbar\, \underbrace{\frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}}}$$

$$\therefore \; |1,0\rangle = \frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}} = \frac{\downarrow_1 \uparrow_2 + \uparrow_1 \downarrow_2}{\sqrt{2}}$$

similarly $\quad S_- |1,0\rangle = \sqrt{2}\hbar\, |1,-1\rangle$

$$\Rightarrow |1,-1\rangle = \frac{1}{\sqrt{2}\hbar} S_- |1,0\rangle$$

Now $\displaystyle S_- |1,0\rangle = S_- \left(\frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} (S_{1-} + S_{2-}) \left[\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2 \right]$

$$= \frac{1}{\sqrt{2}} \left[\hbar \chi_-^1 \chi_-^2 + \hbar\, \chi_-^1 \chi_-^2 \right.$$

$$= \frac{2\hbar}{\sqrt{2}}\, \chi_-^1 \chi_-^2 = \sqrt{2}\hbar\, \chi_-^1 \chi_-^2$$

$$\Rightarrow |1,-1\rangle = \chi_-^1 \chi_-^2 = \downarrow_1 \downarrow_2$$

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finally we can not use S_- to find the singlet state as it is unique and lies outside the triplet states.

$$S_- \left(\frac{\chi_-^1 \chi_+^2 - \chi_+^1 \chi_-^2}{\sqrt{2}} \right) = 0$$