Algebra of angular momentum $\vec{J}$

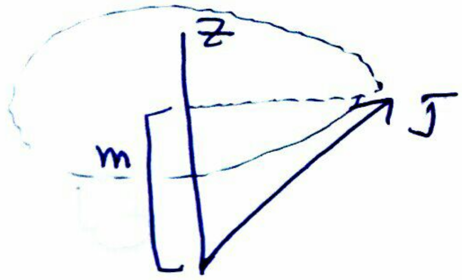
Consider the total angular momentum $\vec{J} = J_x \hat{i} + J_y \hat{j} + J_z \hat{k}$ which satisfy the following commutation relations

$$[J_x, J_y] = i\hbar J_z \,, \quad [J_y, J_z] = i\hbar J_x \,, \quad [J_z, J_x] = i\hbar J_y$$

or shortly $[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$

since J_x, J_y, J_z do not commute, they can't be simultaneously diagonalized (represented by common eigenstates)

- let us take our polar axis as the z-axis



$\Rightarrow$ J_z has a certain value m

$\therefore \langle J_z \rangle = m$

but J_x, J_y don't have certain values

i.e. $\langle J_x \rangle = \langle J_y \rangle = 0$

Now $J^2 = J_x^2 + J_y^2 + J_z^2$ is a scalar operator (positive) $\Rightarrow$ it commutes with J_x, J_y, J_z $\Rightarrow [J^2, J_k] = 0$; $k = x, y, z$

since J^2 commutes with all its components and on the other hand the components J_x, J_y, J_z do not commute, we can choose only one of them to be simultaneously diagonalized with J^2; by convension J_z is chosen. There is nothing special about J_z ; we can just as well take (J^2, J_x) or (J^2, J_y). This mean we can simultaneously know the total angular momentum and one of its components.

$\therefore [J^2, J_z] = 0$

Instead of working with J_x and J_y, it is often easier to define a linear combination of them J_+ and J_-

$$J_+ = J_x + iJ_y \quad ; \quad J_- = J_x - iJ_y = (J_+)^\dagger \quad \text{ladder operators}$$

solving for J_x and J_y (add and subtracte the above two eqn)

$$J_x = \frac{1}{2}(J_+ + J_-) \quad \text{and} \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$J_x^2 = \frac{1}{4}(J_+^2 + J_-^2 + J_+J_- + J_-J_+) \; ; \; J_y^2 = -\frac{1}{4}(J_+^2 + J_-^2 - J_+J_- - J_-J_+)$$

$$\Rightarrow J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

now $J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 + iJ_yJ_x - iJ_xJ_y$

$$= J_x^2 + J_y^2 - i[J_x, J_y] = J_x^2 + J_y^2 + \hbar J_z$$

$$= J^2 - J_z^2 + \hbar J_z$$

similarly $J_-J_+ = J^2 - J_z^2 - \hbar J_z$

one can show that $[J_z, J_\pm] = \pm \hbar J_\pm$ and $[J^2, J_\pm] = 0$

considering just J^2 and J_z as a commuting operators, the the form of the eigen value equations are defined as

$$J^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$$J_z |j,m\rangle = \hbar m |j,m\rangle$$

let us find the optimal values of j, m

now $J^2 = J_x^2 + J_y^2 + J_z^2 \Rightarrow J^2 - J_z^2 = J_x^2 + J_y^2$

$$\therefore (J^2 - J_z^2)|j,m\rangle = J^2|j,m\rangle - J_z^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle - \hbar^2 m^2 |j,m\rangle$$

on the other hand

$$\langle j,m| J^2 - J_z^2 |j,m\rangle = \langle j,m| J_x^2 + J_y^2 |j,m\rangle = \langle j,m| \hbar^2 j(j+1) - \hbar^2 m^2 |j,m\rangle$$

$$= \hbar^2 j(j+1) - \hbar^2 m^2 \langle j,m|j,m\rangle \geq 0$$

Notice that the last equation is always positive ≥ 0

This is because J_x and J_y are Hermitian operators,
$\Rightarrow$ their eigenvalues are real $\Rightarrow$ the sum of squares of the corresponding eigenvalues is ≥ 0

i.e

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \geq 0$$

also notice that
$\langle j,m | j,m \rangle \geq 0$
for square integrable wave functions

$$\therefore \hbar^2 j(j+1) - \hbar^2 m^2 \geq 0$$

$$\Rightarrow \quad m^2 \leq j(j+1)$$

Now the rasing and lowering operators J_+ and J_- work as follow

$$J_+ |j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\ |j,m+1\rangle$$

$$J_- |j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\ |j,m-1\rangle$$

Notice that m is bounded for a given value of j

so we have $\quad J_+ | j, m_{max}\rangle = 0 \quad$ and $\; J_- | j, m_{min}\rangle = 0$

$$\text{so} \quad J_- J_+ | j, m_{max}\rangle = 0$$

$$(J^2 - J_z^2 - \hbar J_z) | j, m_{max}\rangle = 0$$

$$\hbar^2 j(j+1) - \hbar^2 m_{max}^2 - \hbar^2 m_{max} = 0$$

$$j(j+1) = m_{max}^2 + m_{max} = m_{max}(m_{max}+1)$$

$$\Rightarrow \boxed{m_{max} = j}$$

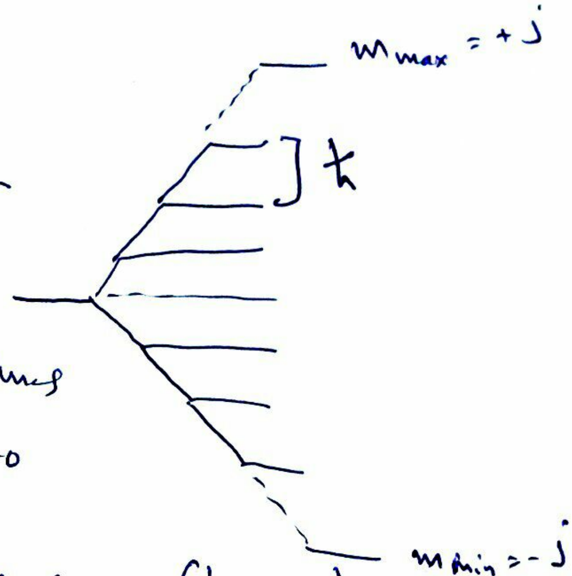
similarly $\quad J_+ J_- | j, m_{min}\rangle = 0$

$$(J^2 - J_z^2 + \hbar J_z) | j, m_{min}\rangle = 0$$

$$\Rightarrow \quad j(j+1) - \hbar^2 m_{min}^2 - \hbar^2 m_{min} = 0$$

$$\Rightarrow \quad j(j+1) = m_{min}(m_{min} - 1) \quad \Rightarrow \quad m_{min} = -j$$

$$\therefore \quad -j \leq m \leq j$$



now Assuming the rugs of the ladder are separated by $\hbar$, then one needs to apply the raising operator $(2J)$ times in order to go from the bottom to the top of the ladder $\Rightarrow$

$$2J = \text{integer} \Rightarrow J = \begin{cases} \text{integer (bosons)} \\ \text{half-integer (fermions)} \end{cases}$$

$$\therefore \quad |\underbrace{j}_{\geq 0}, \underbrace{m}_{(2j+1)\ \text{states}}\rangle$$