Mathematical physics (2)

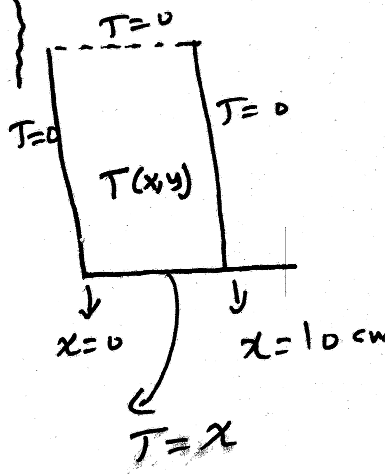
HW #9- solution

Dr. Gassem Alzoubi

problem 13.2.1

(1) Find the steady-state temperature distribution for the semi-infinite plate problem with the bottom edge temperature of $T=x$

- for semi-infinite plate, the solution of $\nabla^2 T=0$ is



$$T(x,y)=\sum_{n=1}^{\infty} b_n\, e^{-\frac{n\pi y}{10}} \sin\left(\frac{n\pi x}{10}\right)$$

now from B.Condition $T(y=0)=x$

$$T(x,0)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$$

$$x=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) \quad \text{, where}$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) \quad ; \quad l=10,\ f(x)=x, \Rightarrow b_n=\frac{2}{10}\int_0^{10} x\sin\left(\frac{n\pi x}{10}\right)dx$$

integrate by parts $u=x$; $dv=\sin\frac{n\pi x}{10}\,dx$

$du=dx$

$$V=-\frac{\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}=\frac{-10}{n\pi}\cos\frac{n\pi x}{10}$$

$$\Rightarrow b_n=\frac{2}{10}\left[\left.\frac{-10x}{n\pi}\cos\frac{n\pi x}{10}\right|_0^{10}+\frac{10}{n\pi}\int_0^{10}\cos\frac{n\pi x}{10}\,dx\right]$$

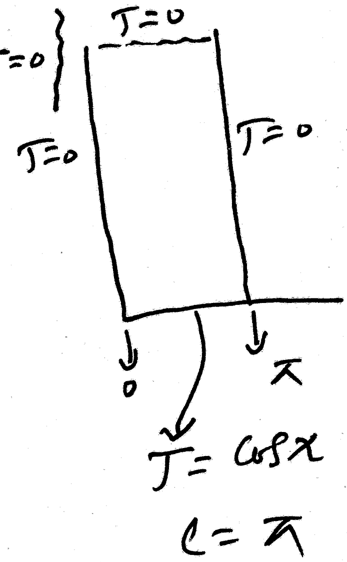
$$=\frac{2}{10}\left\{-\frac{100}{n\pi}(\cos n\pi)+\left(\frac{10}{n\pi}\right)^2\left.\sin\left(\frac{n\pi x}{10}\right)\right|_0^{10}\right\}$$

$$=\frac{2}{10}\left[-\frac{100}{n\pi}(-1)^n+\underbrace{\frac{100}{n^2\pi^2}(\cancel{\sin n\pi}^{\,0}-\cancel{\sin 0}^{\,0})}_{\text{zero}}\right]=-\frac{20}{n\pi}(-1)^n$$

$$=\frac{20}{n\pi}(-1)^{n+1}$$

$$\Rightarrow T(x,y)=\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{n\pi y}{10}}\sin\left(\frac{n\pi x}{10}\right)$$

② problem 13.2.3: Solve the semi-infinite plate problem of bottom width π and held at temperate $T = \cos x$ and the other sides are at $0\,C^\circ$



- for semi-infinite plate, the solution of $\nabla^2 T = 0$ is given by

$$T(x,y) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{n\pi y}{l}} \sin\left(\frac{n\pi x}{l}\right) ; \quad l = \pi$$

$$T(x,y) = \sum_{n=1}^{\infty} b_n\, e^{-ny} \sin nx$$

$$T(x,0) = \cos x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l f(x) \sin nx\, dx ; \quad l = \pi ; \quad f(x) = \cos x$$

$$= \frac{2}{\pi}\int_0^\pi \cos x \sin nx\, dx = \frac{2}{\pi}\left[\frac{n(-1)^n + n}{n^2 - 1}\right]$$

→ use integral calculator www.integral-calculator.com

$$b_n = \begin{cases} 0, & \text{odd } n \\ \dfrac{4n}{\pi(n^2-1)}, & \text{even } n \end{cases}$$

$$\Rightarrow T(x,y) = \frac{4}{\pi}\sum_{\substack{n=2 \\ \text{even } n}}^{\infty} \frac{n}{n^2-1}\, e^{-ny} \sin nx ; \quad n = 2, 4, 6, \ldots$$

Find T at $(x,y) = (\frac{\pi}{2}, 0)$

$$T(\tfrac{\pi}{2}, 0) = \frac{4}{\pi}\sum_{n=2} \frac{n}{n^2-1} \sin\left(\frac{n\pi}{2}\right)$$

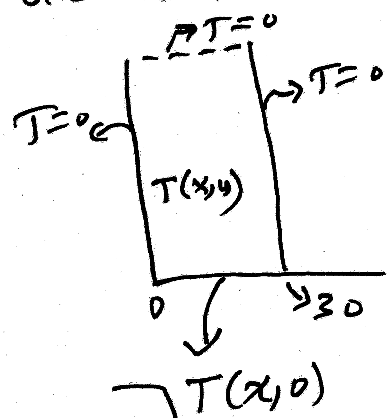
$$= \frac{4}{\pi}\left\{\frac{2}{3}\cancel{\sin\pi}^{\,0} + \frac{4}{15}\cancel{\sin 2\pi}^{\,0} + \frac{6}{35}\cancel{\sin 3\pi}^{\,0} + 0 + \cdots\right\}$$

= zero as expected from $T = \cos x = \cos\frac{\pi}{2} = 0$

③ problem 13.2.4: solve the semi-infinite plate if the bottom edge of width 30 and held at

$$T(x,0) = \begin{cases} x, & 0 < x < 15 \\ 30 - x, & 15 < x < 30 \end{cases}$$, and the other sides are held at $T = 0°$

$$T(x,y) = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n\pi y}{30}} \sin\left(\frac{n\pi x}{30}\right)$$



with $b_n = \frac{2}{30} \int_0^{30} T(x,0) \sin\left(\frac{n\pi x}{30}\right) dx$

$$\Rightarrow b_n = \frac{2}{30}\left[\underbrace{\int_0^{15} x \sin\left(\frac{n\pi x}{30}\right) dx}_{\mathrm{I}} + \underbrace{\int_{15}^{30} (30-x) \sin\left(\frac{n\pi x}{30}\right) dx}_{\mathrm{II}}\right]$$

$$\mathrm{I} = \frac{1}{\pi^2 n^2}\left[900 \sin\left(\frac{n\pi}{2}\right) - 450\, n\pi \cos\left(\frac{n\pi}{2}\right)\right]$$

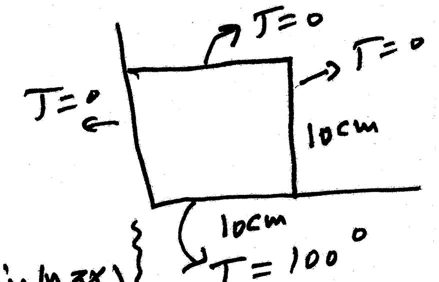
$$\mathrm{II} = \frac{1}{\pi^2 n^2}\left[900 \sin\left(\frac{n\pi}{2}\right) + 450\, n\pi \cos\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow \mathrm{I} + \mathrm{II} = \frac{1800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b_n = \frac{2}{30} \cdot \frac{1800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) = \frac{120}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow T(x,y) = \frac{120}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) e^{-\frac{n\pi}{30} y} \sin\left(\frac{n\pi x}{30}\right)$$

(4) problem 13.2.10: Find the steady-state temperature distribution in a metal plate 10 cm square if one side is held at 100 C° and the other three sides at 0 C°. Find the temperature at the center.



for a rectangular plate (W, H), we found that T(x,y) is given by

$$T(x,y) = \frac{4T_b}{\pi} \sum_{\substack{\text{odd} \\ n}} \left\{ \frac{1}{n \sinh\left(\frac{Hn\pi}{W}\right)} \sinh \frac{n\pi}{W}(H-y) \sin\left(\frac{n\pi x}{W}\right) \right\}$$

Here $W = H = 10$ cm, and $T_b = 100$

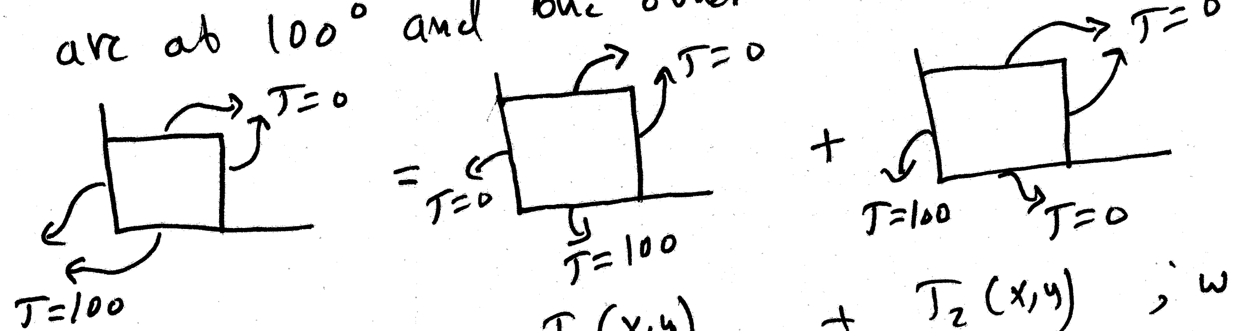
$$T(x,y) = \frac{400}{\pi} \sum_{\substack{\text{odd} \\ n}} \frac{1}{n \sinh(n\pi)} \sinh \frac{n\pi}{10}(10-y) \sin\left(\frac{n\pi x}{10}\right)$$

$$T(5,5) = \frac{400}{\pi} \sum_{\substack{\text{odd} \\ n}} \frac{1}{n \sinh(n\pi)} \sinh\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \text{ ; calculate few terms}$$

$$\approx 25.36 - 0.38 + 0.009 - \cdots \approx 25 C°$$

(5) problem 13.2.11: Find the steady state temperature distribution of last problem (13.2.10) if two adjacent sides are at 100° and the other two at 0°.

L = H = 10cm



$$T(x,y) = T_1(x,y) + T_2(x,y) \text{ ; where}$$

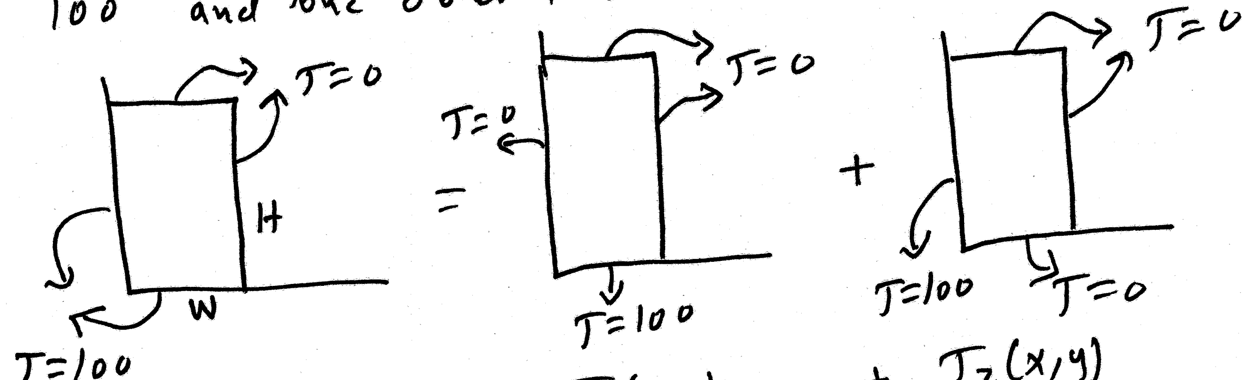
$$T_1(x,y) = \frac{400}{\pi} \sum_{\substack{\text{odd} \\ n}} \frac{1}{n \sinh n\pi} \sinh \frac{n\pi}{10}(10-y) \sin\left(\frac{n\pi x}{10}\right)$$

to get $T_2(x,y)$, replace x by y ⇒

$$T_2(x,y) = \frac{400}{\pi} \sum_{\substack{\text{odd} \\ n}} \frac{1}{n \sinh n\pi} \sinh \frac{n\pi}{10}(10-x) \sin\left(\frac{n\pi y}{10}\right)$$

$$\Rightarrow T(x,y) = T_1(x,y) + T_2(x,y)$$

⑥ problem 13.2.12: Find $T(x,y)$ in a rectangular plate $W = 10\,cm$ and $H = 30\,cm$ if two adjacent sides are held at $100°$ and the other two sides at $0°$.



$$T(x,y) = T_1(x,y) + T_2(x,y)$$

using $$T(x,y) = \frac{4T_b}{\pi} \sum_{\text{odd } n} \frac{1}{n \sinh\left(\frac{Hn\pi}{W}\right)} \sinh \frac{n\pi}{W}(H-y) \sin\left(\frac{n\pi x}{W}\right)$$

$\Rightarrow$

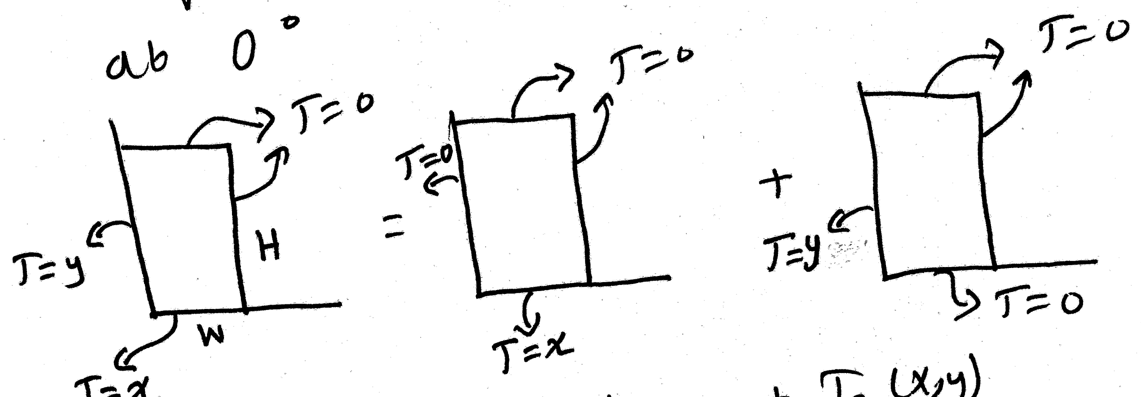
$$T_1(x,y) = \frac{400}{\pi} \sum_{\text{odd } n} \frac{1}{n \sinh(3n\pi)} \sinh \frac{n\pi}{10}(30-y) \sin\left(\frac{n\pi x}{10}\right)$$

Now to get $T_2(x,y)$, replace x by y and exchange W and H

$$T_2(x,y) = \frac{400}{\pi} \sum_{\text{odd } n} \frac{1}{n \sinh\left(\frac{n\pi}{3}\right)} \sinh \frac{n\pi}{30}(10-x) \sin\left(\frac{n\pi y}{30}\right)$$

$$\therefore\ T(x,y) = T_1(x,y) + T_2(x,y)$$

⑦ problem 13.2.13: Find the steady-state temperature distribution in a rectangular plate $W = 10$, $H = 20\,cm$, if the two adjacent sides along the axes are held at temperatures $T = x$ and $T = y$ and the other two sides at $0°$



$$T(x,y) = T_1(x,y) + T_2(x,y)$$

Now $T_1(x,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{10}(20-y) \sin\left(\frac{n\pi x}{10}\right)$, where

$$B_n = \frac{b_n}{\sinh\left(\frac{H}{W} n\pi\right)} = \frac{b_n}{\sinh(2\pi n)}, \text{ with}$$

$$b_n = \frac{2}{W}\int_0^W f(x) \sin\left(\frac{n\pi x}{W}\right) dx \text{ ; with } f(x) = T = x, \quad W = 10$$

$$= \frac{2}{10}\int_0^{10} x \sin\left(\frac{n\pi x}{10}\right) dx = \frac{20}{n\pi}(-1)^{n+1}$$

$$\Rightarrow B_n = \frac{20}{n\pi} \frac{(-1)^{n+1}}{\sinh(2\pi n)}$$

$$\Rightarrow T_1(x,y) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(2\pi n)} \sinh \frac{n\pi}{10}(20-y) \sin\left(\frac{n\pi x}{10}\right)$$

to obtain $T_2(x,y)$, we need to recalculate B_n

$$B_n = \frac{b_n}{\sinh\left(\frac{n\pi}{2}\right)} \text{ ; } b_n = \frac{2}{H}\int_0^H f(y) \sin\left(\frac{n\pi y}{H}\right) dy \text{ ; } H = 20, \; f(y) = T = y$$

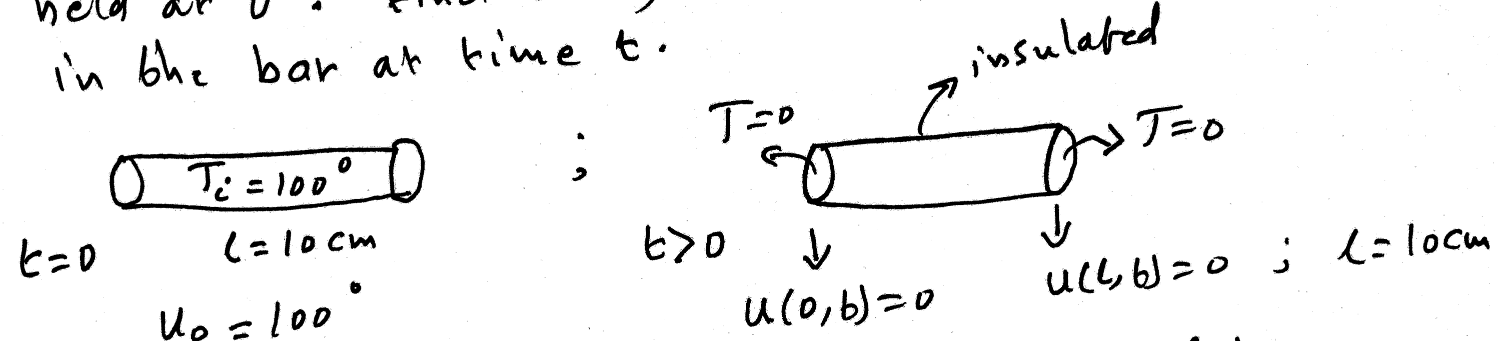
$$b_n = \frac{2}{20}\int_0^{20} y \sin\left(\frac{n\pi y}{20}\right) dy = \frac{40}{n\pi}(-1)^{n+1}$$

$$\Rightarrow B_n = \frac{40}{n\pi} \frac{(-1)^{n+1}}{\sinh\left(\frac{n\pi}{2}\right)}$$, Now to get $T_2(x,y)$ replace x by y and exchange W and H and use the new B_n

$$\Rightarrow T_2(x,y) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh\left(\frac{n\pi}{2}\right)} \sinh \frac{n\pi}{20}(10-x) \sin\left(\frac{n\pi y}{10}\right)$$

finally $T(x,y) = T_1(x,y) + T_2(x,y)$

⑧ problem 13.3.2! a bar 10cm long with insulated sides initially at $T = 100°$. starting at $t=0$, the ends are held at $0°$. Find $T(x,y)$ the temperature distribution in the bar at time t.



solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$; let $u(x,t) = T(t)\,X(x)$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha^2}\frac{1}{T}\frac{dT}{dt} = -k^2$$

$$\Rightarrow \frac{dT}{dt} = -k^2\alpha^2 T \Rightarrow \frac{dT}{T} = -k^2\alpha^2 t \Rightarrow T(t) = e^{-k^2\alpha^2 t}$$

and

$$X'' + k^2 X = 0 \Rightarrow X(t) = \begin{cases} \sin kx \ ✓ \\ \cos kx \ ✗ \end{cases}$$

but from Boundary condition $u(0,t)=0$, the $\cos kx$ can't be satisfied, so it is excluded.

$$\Rightarrow u(x,t) = e^{-k^2\alpha^2 t}\sin kx \ ; \text{ from the boundary condition}$$

$u(L,0)=0 \Rightarrow \sin kl = 0 \Rightarrow kl = n\pi \Rightarrow k_n = \dfrac{n\pi}{l}$
$n = 1, 2, 3, ...$

$$\Rightarrow u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\sin\frac{n\pi x}{l} \ ; \ l = 10\text{ cm}$$

the most general solution is a linear combination of $u_n(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)$$

the b_n is found using the initial condition at $t=0$

$$u(x,0) = u_0 = 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad ; \quad f(x) = 100$$

$$= \frac{2}{l}\int_0^l (100)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{200}{l}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{200}{l}\,\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\Bigg|_0^l = -\frac{200}{n\pi}\left[\cos n\pi - 1\right]$$

$$= -\frac{200}{n\pi}\left\{(-1)^n - 1\right\} = \frac{200}{n\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0, & \text{even } n \\ \frac{400}{n\pi}, & \text{odd } n \end{cases}$$

$$\Rightarrow u(x,t) = \frac{400}{\pi}\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi\alpha}{10}\right)^2 t} \sin\left(\frac{n\pi x}{10}\right)$$

Note that as $t \to \infty$, the final steady-state temperature is zero.

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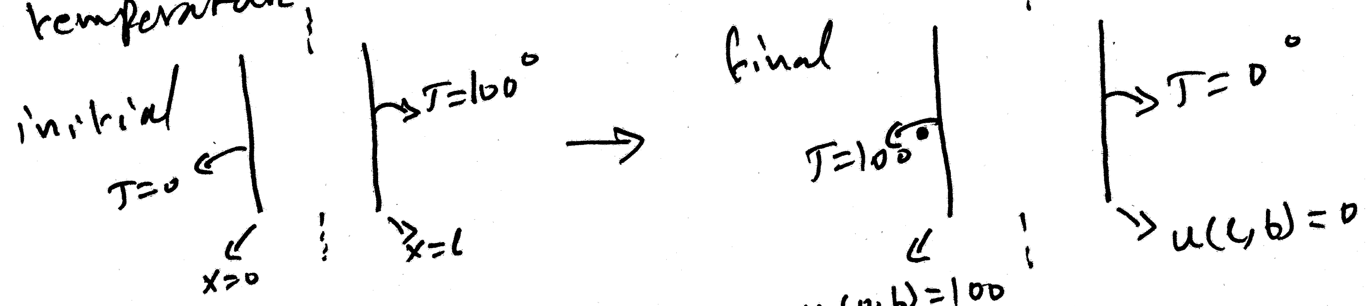
(9) <u>problem 13.3.3:</u> in the initial steady-state of an infinite slab of thickness l, the face $x=0$ is at $0°$ and the face at $x=l$ is at $100°$. From $t=0$ on, the $x=0$ face is held at $100°$ and the $x=l$ face at $0°$. Find the temperature distribution at a time t.



$$\Rightarrow u_0 = \frac{100}{l}x \qquad u_f = 100 - \frac{100}{l}x$$

$$u(x,t) = u_f + \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$

$$= 100 - \frac{100}{l}x + \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$

from initial condition $u(x,0) = u_0 = \frac{100}{l}x$, we get

$$u(x,0) = \frac{100}{l}x = 100 - \frac{100}{l}x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \quad \frac{200}{l}x - 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{, with}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l \left(\frac{200}{l}x - 100\right) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{400}{l^2}\int_0^l x \sin\left(\frac{n\pi x}{l}\right)dx - \frac{200}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{400}{n\pi}(-1)^{n+1} - \left[-\frac{200}{n\pi}\left((-1)^n - 1\right)\right]$$

$$= \frac{400}{n\pi}(-1)^{n+1} + \frac{200}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{odd } n \\ -\frac{400}{n\pi}, & \text{even } n \end{cases}$$

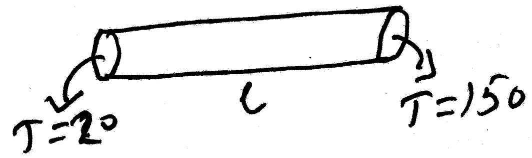
$$\Rightarrow u(x,t) = 100 - \frac{100}{l}x - \frac{400}{\pi}\sum_{\substack{n=2 \\ \text{even } n}}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$

Note that as $t \to \infty$, $u \to u_f = 100 - \frac{100}{l}x$ as expected

$\downarrow$ final steady state temperature distribution

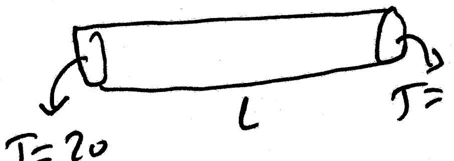
(10) problem 13.3.6: show that the following problem is easily solved using eq^n (3.15): The ends of a bar are initially at 20° and 150°. at $t=0$ on, the 150° end is changed to 50°. Find the time dependent temperature distribution

initial $t=0$



$$u_0 = \frac{130}{l}x + 20$$
$$= \frac{100}{l}x + \left(\frac{30}{l}x + 20\right)$$

final $t>0$



$$u_f = \frac{30}{l}x + 20$$
$$= 0 + \left(\frac{30}{l}x + 20\right)$$

$$\therefore \quad u_0 = \boxed{\frac{100x}{l}} + \left(\frac{30}{l}x + 20\right)$$
$$u_f = \boxed{0} + \left(\frac{30}{l}x + 20\right)$$

original problem as described in textbook or in our lecture notes

We see that the same linear term has been added to both u_0 and u_f $\Rightarrow$ so the Fourier series will not change, so the solution is given by

$$u(x,t) = u_f + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{30x}{l} + 20 + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$

This can be proven from

$$u(x,0) = u_0 = u_f + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{130x}{l} + 20 = \frac{30x}{l} + 20 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

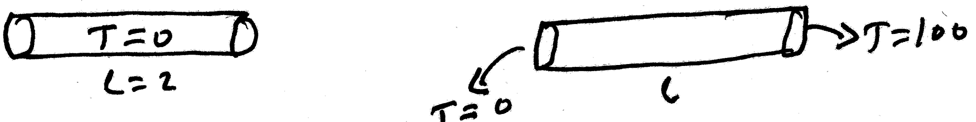
$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{200}{n\pi}(-1)^{n-1}$$

same as the original problem

(11) problem 13.3.8: a bar of length 2 ($l=2$) is initially at $0°$. From $t=0$ on, the $x=0$ end is held at $0°$ and the $x=2$ end at $100°$. Find the time-dependent temperature distribution.



$u(x,0)=u_0=0$ initial

$u_f = 50x$

the general solution of zero Boundary conditions of the final steady-state ($u(0,t)=u(l,t)=0$) $\to u_f=0$ is given by

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)$$

; but here the final steady state is not zero; it is $u_f = 50x \Rightarrow$ so we just need to add u_f to the last equation $\Rightarrow$

$$u(x,t)=u_f+\sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)=50x+\sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)$$

from initial condition

$$u(x,0)=u_0=0=50x+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

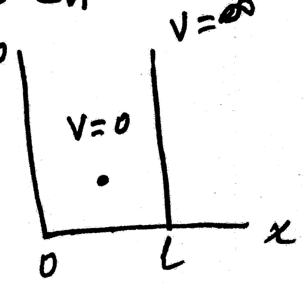
$$\Rightarrow -50x=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \Rightarrow b_n=\frac{2}{l}\int_0^l(-50x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow l=2 \Rightarrow b_n=\int_0^2(-50x)\sin\left(\frac{n\pi x}{2}\right)dx=-50\underbrace{\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx}_{\frac{4}{n^2\pi^2}\left[\underbrace{\sin(n\pi)}_{0}-\pi n\underbrace{\cos n\pi}_{(-1)^n}\right]}$$

$$b_n=-50\left(\frac{-4}{n\pi}(-1)^n\right)=\frac{200}{n\pi}(-1)^n$$

$$\Rightarrow u(x,t)=50x+\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\left(\frac{n\pi\alpha}{2}\right)^2 t}\sin\left(\frac{n\pi x}{2}\right)$$

(12) Solve the particle in a box problem to find $\Psi(x,t)$ if
(13.3.11) $\Psi(x,0) = 1$ on $(0,\pi)$. Find the eigenvalues E_n



$$H\psi = i\hbar \frac{\partial \psi}{\partial t}, \quad H = \frac{p^2}{2m} + V(x) \; ; \; V(x) = 0$$

$$= \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \; ; \; p = \frac{\hbar}{i}\frac{d}{dx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x,t)}{dx^2} = i\hbar\frac{\partial \psi(x,t)}{\partial t} \; ; \quad \text{let } \psi(x,t) = T(t)\psi(x)$$

$$-\frac{\hbar^2}{2m}T(t)\frac{d^2\psi(x)}{dx^2} = i\hbar\,\psi(x)\frac{dT}{dt} \quad ; \text{ divide by } T(t)\psi(x)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = i\hbar\frac{1}{T}\frac{dT}{dt} = \text{constant} = E$$

$$\Rightarrow i\hbar\frac{1}{T}\frac{dT}{dt} = E \Rightarrow \frac{dT}{T} = -\frac{iE}{\hbar}dt \Rightarrow T(t) = e^{-\frac{iE}{\hbar}t}$$

and
$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = E \Rightarrow \frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$$\psi''(x) + k^2\psi(x) = 0 \; ; \; k^2 = \frac{2mE}{\hbar^2}$$

$$\Rightarrow \psi(x) = \begin{cases} \sin kx \\ \cos kx \end{cases}$$

$$\Rightarrow \psi(x,t) = T(t)\psi(x) = e^{-\frac{iE}{\hbar}t}\begin{cases} \sin kx \\ \cos kx \end{cases}$$

at $x=0$ $\psi(0,t)$ must vanish, so $\cos kx$ is excluded.

$$\Rightarrow \psi(x,t) = e^{-\frac{iE}{\hbar}t}\sin kx$$

and $\psi(L,t) = 0 = \underbrace{e^{-\frac{iE}{\hbar}t}}_{\neq 0}\sin kL = 0 \Rightarrow \sin kL = 0$

$$\Rightarrow kL = n\pi \Rightarrow k_n = \frac{n\pi}{L}$$

$n = 1, 2, 3, \ldots$

$$\therefore \psi_n(x,t) = e^{-\frac{iE_n t}{\hbar}}\sin\frac{n\pi x}{L} \; ; \; E_n = \frac{\hbar^2 k_n^2}{2m}$$

$$= \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

since schrodinger equation is linear $\Rightarrow$ the general solution is $\Psi(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{iE_n t}{\hbar}} \sin \frac{n\pi x}{l}$

b_n is found from initial condition, $\Psi(x,0) = 1$ on $(0,\pi)$

$\Rightarrow \Psi(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, $\Rightarrow b_n = \frac{2}{l}\int_0^l \Psi(x,0) \sin \frac{n\pi x}{l} dx$

$l = \pi \Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$

$= \frac{-2}{n\pi}\left[\cos n\pi - \cos 0\right] = -\frac{2}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{n\pi}\left[1-(-1)^n\right]$

$= \begin{cases} 0, & \text{even } n \\ \frac{4}{n\pi}, & \text{odd } n \end{cases}$; $E_n = \frac{\hbar^2 n^2 \pi^2}{2ml^2} = \frac{\hbar^2 n^2 \pi^2}{2m\pi^2} = \frac{\hbar^2 n^2}{2m}$

$\Rightarrow \Psi(x,t) = \frac{4}{\pi}\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{n} e^{-i\frac{E_n t}{\hbar}} \sin nx$

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(13) problem 13.3.12 : Do the last problem with $\Psi(x,0) = \sin^2 \pi x$ on $(0,1)$

$\Psi(x,t) = \sum_{n=1}^{\infty} b_n e^{-i\frac{E_n t}{\hbar}} \sin \frac{n\pi x}{l}$;

$b_n = \frac{2}{l}\int_0^l \Psi(x,0) \sin \frac{n\pi x}{l} dx$; $l = 1$

$= 2\int_0^1 \sin^2 \pi x \sin n\pi x \, dx = \frac{4\left[(-1)^n - 1\right]}{\pi n(n^2-4)}$

$= \begin{cases} 0, & \text{even } n \\ \frac{-8}{n\pi(n^2-4)} = \frac{8}{n\pi(4-n^2)}, & \text{odd } n \end{cases}$; $E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ml^2}$

$l = 1 \Rightarrow \quad = \frac{\hbar^2 \pi^2 n^2}{2m}$

$\Psi(x,t) = \frac{8}{\pi}\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n(4-n^2)} e^{-\frac{iE_n t}{\hbar}} \sin n\pi x$