Mathematical Physics (2)
HW # 8 – solution
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① problem 12.22.7: prove that the functions $H_n(x)$ are orthogonal on $(-\infty, \infty)$ with respect to the weight function e^{-x^2}

the $H_n(x)$ polynomials satisfy Hermite equation

$$y'' - 2xy' + 2ny = 0 \Rightarrow$$

$$H_n'' - 2xH_n' + 2nH_n = 0 \quad \text{--- (1)}$$

$$H_m'' - 2xH_m' + 2mH_m = 0 \quad \text{--- (2)}$$

multiply (1) by H_m and (2) by H_n and subtract, we get

$$[H_n'' H_m - H_m'' H_n] - 2x[H_n' H_m - H_m' H_n] + 2(n-m) H_n H_m = 0 \quad \text{--- (3)}$$

this can be written as

$$\frac{d}{dx}[H_n' H_m - H_m' H_n] - 2x[H_n' H_m - H_m' H_n] = 2(m-n) H_n H_m \quad \text{...(4)}$$

multiply by e^{-x^2}

$$e^{-x^2}\frac{d}{dx}[H_n' H_m - H_m' H_n] - 2xe^{-x^2}[H_n' H_m - H_m' H_n] = 2(m-n)e^{-x^2} H_n H_m$$

combine the first two terms

$$\frac{d}{dx}\left[e^{-x^2}(H_n' H_m - H_m' H_n)\right] = 2(m-n)e^{-x^2} H_n H_m$$

integrate

$$\int_{-\infty}^{\infty} \frac{d}{dx}\left[e^{-x^2}(H_n' H_m - H_m' H_n)\right] dx = 2(m-n)\int_{-\infty}^{\infty} e^{-x^2} H_n H_m \, dx$$

$$\underbrace{e^{-x^2}(H_n' H_m - H_m' H_n)\Big|_{-\infty}^{\infty}}_{\text{zero}} = 2(m-n)\int_{-\infty}^{\infty} e^{-x^2} H_n H_m \, dx$$

$\Rightarrow$ since $m \neq n$ $\Rightarrow$ $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0$

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② problem 12.22.9: use the generating function of $H_n(x)$ to derive the recursion relation $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$

$$e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{H_n}{n!} h^n \text{ ; Differentiate both sides w.r.t. } h$$

$$(2x-2h)\, e^{+2xh-h^2} = \sum_{n=0}^{\infty} \frac{H_n}{n!} n h^{n-1}$$

$$(2x-2h)\sum_{n=0}^{\infty} \frac{H_n}{n!} h^n = \sum_{n=0}^{\infty} \frac{H_n}{n!} n h^{n-1}$$

$$2x\sum_0 \frac{H_n}{n!} h^n - 2\sum_0 \frac{H_n}{n!} h^{n+1} = \sum_0 \frac{H_n}{n!} n h^{n-1} \text{ ; unify powers to } h^n$$

$$2x\sum_0 \frac{H_n}{n!} h^n - 2\sum_1 \frac{H_{n-1}}{(n-1)!} h^n = \sum_{-1} \frac{H_{n+1}}{(n+1)!}(n+1) h^n$$

$$2x\frac{H_0}{0!} + \sum_1 \frac{2xH_n}{n!} h^n - 2\sum_1 \frac{H_{n-1}}{(n-1)!} h^n = \frac{H_0}{1!} + \sum_1 \frac{H_{n+1}}{(n+1)!}(n+1) h^n$$

now using $n! = n(n-1)!$ and $(n+1)! = (n+1)\,n!$, we get

$$2xH_0 + \sum_1 \frac{2xH_n}{n!} h^n - \sum_1 \frac{2n H_{n-1}}{n!} h^n = H_0 + \sum_1 \frac{H_{n+1}}{n!} h^n$$

equate coefficients of h^n

$$2x\frac{H_n}{\cancel{n!}} - \frac{2n H_{n-1}}{\cancel{n!}} = \frac{H_{n+1}}{\cancel{n!}} \Rightarrow \boxed{H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)}$$

③ **problem:** use the generating function of $H_n(x)$ to prove that $H_n(-x) = (-1)^n H_n(x)$

$$\Phi(x,h) = e^{2xh - h^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n \quad , \text{ now let } x \to -x, \text{ and } h \to -h$$

$$\Phi(-x,-h) = \sum_{n=0}^{\infty} \frac{H_n(-x)}{n!} (-h)^n = \sum_{n=0}^{\infty} (-1)^n \frac{H_n(-x)\, h^n}{n!}$$

but $\Phi(-x,-h) = \Phi(x,h)$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n = \sum_{n=0}^{\infty} \frac{(-1)^n H_n(-x)\, h^n}{n!}$$

$$\Rightarrow H_n(x) = (-1)^n H_n(-x) \quad \text{or} \quad H_n(-x) = (-1)^n H_n(x)$$

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④ **problem;** evaluate $\int_{-\infty}^{\infty} e^{-x^2} x H_n(x) H_m(x)\, dx$

using the recursion relation 22.17 (b)

$x H_n(x) = \frac{1}{2} H_{n+1}(x) + n H_{n-1}(x)$, we get

$$\int_{-\infty}^{\infty} e^{-x^2} [x H_n(x)] H_m(x)\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} H_{n+1} H_m\, dx + n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1} H_m\, dx$$

$$= \frac{1}{2} \sqrt{\pi}\, 2^m m!\, \delta_{n+1,m} + n \sqrt{\pi}\, 2^m m!\, \delta_{n-1,m}$$

$$= \sqrt{\pi}\, 2^m m! \left[\frac{1}{2} \delta_{n+1,m} + n\, \delta_{n-1,m} \right]$$

(5) <u>problem:</u> evaluate $\int_{-\infty}^{\infty} x^2 H_n^2 e^{-x^2} dx$

using the recursion relation $xH_n = \frac{1}{2}[H_{n+1} + 2n H_{n-1}]$

we have $x^2 H_n^2 = \frac{1}{4}[H_{n+1} + 2n H_{n-1}]^2$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 H_n^2 e^{-x^2} dx = \frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}^2 dx$$

$$+ n^2 \int_{-\infty}^{\infty} H_{n-1}^2 e^{-x^2} dx + n \underbrace{\int_{-\infty}^{\infty} H_{n+1} H_{n-1} e^{-x^2} dx}_{\text{zero as } H_{n+1} \text{ and } H_{n-1} \text{ are orthogonal}}$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 H_n^2 e^{-x^2} dx = \frac{1}{4}\sqrt{\pi}\, 2^{n+1} (n+1)! + n^2 \sqrt{\pi}\, 2^{n-1} (n-1)!$$

$$= \frac{1}{4}\sqrt{\pi}\, 2^{n+1} (n+1)\, n! + n^2 \sqrt{\pi}\, 2^{n-1} \frac{n!}{n}$$

$$= \sqrt{\pi}\, 2^{n-1} (n+1)\, n! + n \sqrt{\pi}\, 2^{n-1} n!$$

$$= \sqrt{\pi}\, 2^{n-1} n! [n+1+n]$$

$$= \sqrt{\pi}\, 2^{n-1} n! [2n+1]$$

(6) problem 12.22.11 Harmonic oscillator

we know that Hermite Polynomials satisfy Hermite equation $H_n'' - 2x H_n' + 2n H_n = 0$. for quantum harmonic oscillator, the wavefunction must vanish as $x \to \pm\infty$. The trial wave function takes the form $\psi_n(x) = e^{-x^2/2} H_n(x) \Rightarrow H_n(x) = e^{x^2/2} \psi_n(x)$. substitute in Hermite equation yields,

$$H_n' = e^{x^2/2} \psi_n' + x e^{x^2/2} \psi_n$$

$$H_n'' = e^{x^2/2} \psi_n'' + 2x e^{x^2/2} \psi_n' + e^{x^2/2} \psi_n + x^2 e^{x^2/2} \psi_n$$

substitute back in Hermite equation yields

$$\psi_n''(x) + [(2n+1) - x^2] \psi_n(x) = 0 \quad \text{--- (1)}$$

- for one dimensional quantum harmonic oscillator

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

when $p = \frac{\hbar}{i}\frac{d}{dx}$

$$\Rightarrow H\psi = E\psi \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x)$$

rearrange

$$\psi''(x) + \left[\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}\right]\psi(x) = 0$$

let $y = \alpha x$; $\alpha^4 = \frac{m^2\omega^2}{\hbar^2}$

and let $y = \alpha x \Rightarrow dy = \alpha dx \Rightarrow \frac{d}{dy} = \frac{1}{\alpha}\frac{d}{dx}$

$\Rightarrow \frac{d}{dx} = \alpha \frac{d}{dy}$; $\frac{d^2}{dx^2} = \alpha^2 \frac{d^2}{dy^2}$

$$\Rightarrow \alpha^2 \frac{d^2\psi(y)}{dy^2} + \left[\frac{2mE}{\hbar^2} - \alpha^4 \frac{y^2}{\alpha^2}\right]\psi(y) = 0$$

$$\psi''(y) + \left[\frac{2mE}{\alpha^2\hbar^2} - y^2\right]\psi(y) = 0 \quad \Rightarrow$$

$$\boxed{\psi''(y) + \left[\frac{2E}{\hbar\omega} - y^2\right]\psi(y) = 0} \quad \text{---} (2)$$

compare with equation (1), we have

$E = \hbar\omega(n + 1/2)$; $n = 0, 1, 2, 3, ---$

$\frac{2E}{\hbar\omega} = 2n+1 \Rightarrow$

and $\psi_n(y) = e^{-y^2/2} H_n(y)$; let us normalize this wave function

$$\int_{-\infty}^{\infty} |\psi_n(y)|^2 dy = \alpha \underbrace{\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx}_{\text{we want this to be } = 1} \quad ; \quad dy = \alpha dx$$

$\Rightarrow \int_{-\infty}^{\infty} |\psi_n(y)|^2 dy = \alpha \Rightarrow$ when $\psi_n(y) = \underset{\text{normalization constant}}{c}\, e^{-y^2/2} H_n(y)$

$$c^2 \int_{-\infty}^{\infty} e^{-y^2} H_n^2(y)\, dy = \alpha \Rightarrow c^2 \sqrt{\pi}\, 2^n n! = \alpha$$

$$\Rightarrow c = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \Rightarrow \psi_n(y) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-y^2/2} H_n(y)$$

$$\text{or } \psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right)$$