Mathematical physics (2)
HW #7 – solution
Dr. Gassem Alzoubi

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1) prove recursion relation 15.5 in textbook

$$J_p'(x) = \underbrace{\frac{p}{x} J_p(x) - J_{p+1}(x)}_{\text{L.H.S}} = \underbrace{-\frac{p}{x} J_p(x) + J_{p-1}(x)}_{\text{R.H.S}}$$

a) let us prove L.H.S first

$$\frac{d}{dx}\left(J_p(x)\right) = \frac{d}{dx}\left[x^{p}\left(x^{-p} J_p\right)\right]$$

$$= p x^{+p-1}\left(x^{-p} J_p\right) + x^{p} \underbrace{\frac{d}{dx}\left(x^{-p} J_p\right)}_{= -x^{-p} J_{p+1} \text{ using } 15.2}$$

$$= \frac{p}{x} J_p - x^{p} x^{-p} J_{p+1}$$

$$= \frac{p}{x} J_p(x) - J_{p+1}(x) \checkmark$$

b) $$\frac{d}{dx}\left(J_p(x)\right) = \frac{d}{dx}\left[x^{-p}\left(x^{p} J_p\right)\right]$$

$$= -p x^{-p-1}\left(x^{p} J_p\right) + x^{-p} \underbrace{\frac{d}{dx}\left(x^{p} J_p\right)}_{= x^{p} J_{p-1} \text{ using } 15.1}$$

$$= -\frac{p}{x} J_p + x^{-p} x^{p} J_{p-1}$$

$$= -\frac{p}{x} J_p(x) + J_{p-1}(x) \checkmark$$

see that, for $p = 1$, $J_1'(x) = J_0 - \frac{J_1}{x}$; for $p = 2$ $J_2'(x) = -\frac{2}{x} J_2 + J_1$

② problem 12.15.7 (a) show that $\int_0^\infty J_1(x)\,dx = 1$

using recursion relation 15.5 $J_p'(x) = \frac{p}{x} J_p - J_{p+1}$

set $p=0 \Rightarrow J_0'(x) = -J_1(x) \Rightarrow J_1(x) = -J_0'(x)$, so

$$\int_0^\infty J_1(x)\,dx = -\int_0^\infty J_0'(x)\,dx = -J_0(x)\Big|_0^\infty = -[J_0(\infty) - J_0(0)]$$
$$= -[0-1] = +1$$

we can also show that

$\int_0^\infty J_0(x)\,dx = 1$; using recursion relation 15.5

$$\begin{cases} J_p' = -\frac{p}{x} J_p + J_{p-1}, & \text{set } p=1 \\ J_1' = -\frac{J_1}{x} + J_0 \Rightarrow J_0 = J_1' + \frac{1}{x} J_1 \end{cases}$$

$$\int_0^\infty \left[J_1' + \frac{1}{x} J_1\right] dx$$

$$= \int_0^\infty J_1'(x)\,dx + \int_0^\infty x^{-1} J_1(x)\,dx = J_1(x)\Big|_0^\infty + \int_0^\infty x^{-1} J_1(x)\,dx$$

$$= [\cancel{J_1(\infty)}^{0} - \cancel{J_1(0)}^{0}] + \int_0^\infty x^{-1} J_1(x)\,dx \text{ ; using } \int_0^\infty x^{-1} J_p(x)\,dx = \frac{1}{p}$$

$$= 0 + 1 = 1$$

see lecture notes for proof

$$\int_0^\infty x^{-1} J_1(x) = \frac{1}{1} = 1$$

③ problem 12.15.8: show that

$$\int_0^\infty J_1(x)\,dx = \int_0^\infty J_3(x)\,dx = \cdots = \int_0^\infty J_{2n+1}(x)\,dx$$

and

$$\int_0^\infty J_0(x)\,dx = \int_0^\infty J_2(x)\,dx = \cdots = \int_0^\infty J_{2n}(x)\,dx$$

let us start with odd $p = 2n+1$, and using the recursion relation 15.4, we have

$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x) \quad ; \quad p = 2n+1$

$J_{2n} - J_{2n+2} = 2J_{2n+1}' \Rightarrow J_{2n+2} = J_{2n} - 2J_{2n+1}'$

integrate both sides on $(0,\infty)$

$$\int_0^\infty J_{2n+2}(x)\,dx = \int_0^\infty J_{2n}(x)\,dx - 2\int_0^\infty J_{2n+1}'(x)\,dx$$

$$= \int_0^\infty J_{2n}(x)\,dx - 2\underbrace{J_{2n+1}(x)\Big|_0^\infty}_{J_{2n+1}(\infty) - J_{2n+1}(0)} \quad ; \quad n = 0,1,2,3\ldots$$

$J_{2n+1}(\infty) - J_{2n+1}(0)$

$0 \quad - 0 = 0$

$$= \int_0^\infty J_{2n}(x)\,dx$$

$\Rightarrow n=0 \Rightarrow \int_0^\infty J_0(x)\,dx = \int_0^\infty J_2(x)\,dx$

$n=1 \Rightarrow \int_0^\infty J_2(x)\,dx = \int_0^\infty J_4(x)\,dx$

$n=2 \Rightarrow \int_0^\infty J_4(x)\,dx = \int_0^\infty J_6(x)\,dx, \ \ldots$ and so on

$$\Rightarrow \int_0^\infty J_0(x)\,dx = \int_0^\infty J_2(x)\,dx = \cdots = \int_0^\infty J_{2n}(x)\,dx$$

- for even $p = 2n \ ; \ n > 0 \Rightarrow J_{2n-1} - J_{2n+1} = 2J_{2n}'$

integrate

$$\int_0^\infty J_{2n+1}(x)\,dx = \int_0^\infty J_{2n-1}(x)\,dx - 2\int_0^\infty J_{2n}'(x)\,dx$$

$$= \int_0^\infty J_{2n-1}(x)\,dx - 2\cancel{J_{2n}(x)}\Big|_0^\infty$$

$$= \int_0^\infty J_{2n-1}(x)\,dx$$

$$\Rightarrow n=1 \quad \int_0^\infty J_1(x)\,dx = \int_0^\infty J_3(x)\,dx$$

$$n=2, \quad \int_0^\infty J_3(x)\,dx = \int_0^\infty J_5(x)\,dx$$

$$n=3, \quad \int_0^\infty J_5(x)\,dx = \int_0^\infty J_7(x)\,dx, \ \ldots \text{ and so on}$$

$$\Rightarrow \int_0^\infty J_1(x)\,dx = \int_0^\infty J_3(x)\,dx = \cdots = \int_0^\infty J_{2n+1}(x)\,dx$$

but we already found that

$$\int_0^\infty J_0(x)\,dx = 1 \quad \text{and} \quad \int_0^\infty J_1(x)\,dx = 1, \text{ so}$$

$$\Rightarrow \int_0^\infty J_n(x)\,dx = 1 \quad \text{for any } n$$

<u>problem 4</u>: find $\int x^{-3} J_5(x)\,dx$

$$\Rightarrow \int x^{-3} J_5(x)\,dx = \int x^2 (x^{-4} J_5)\,dx \quad ; \text{ integrate by parts}$$

let $u = x^2$, $du = 2x\,dx$; $dv = x^{-4} J_5\,dx$, $v = \underbrace{\int x^{-4} J_5\,dx}_{= -x^{-4} J_4}$; use $\int x^{-p} J_{p+1}\,dx = -x^{-p} J_p + C$

$$\Rightarrow \int x^{-3} J_5\,dx = x^2(-x^{-4} J_4) + 2\int x^{-3} J_3\,dx = -x^{-2} J_4 + 2\left[-x^{-3} J_3\right] + C$$

$$= -x^{-2} J_4 - 2x^{-3} J_3 + C$$

⑤ problem 12.16.2! solve $y'' + 4x^2 y = 0$

compare with $y'' + \frac{1-2a}{x} y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2} \right] y = 0$

with solution of the form $y(x) = x^a Z_p(bx^c)$

$\Rightarrow 1-2a = 0 \Rightarrow 1 = 2a \Rightarrow \boxed{a = 1/2}$,

$b^2c^2x^{2c-2} = 4x^2 \Rightarrow 2c-2 = 2 \Rightarrow 2c = 4 \Rightarrow \boxed{c=2}$, and

$b^2c^2 = 4 \Rightarrow bc = 2 \Rightarrow b = \frac{2}{c} = \frac{2}{2} = 1$ ✓

and $a^2 - p^2c^2 = 0 \Rightarrow p = \frac{a}{c} = \frac{\frac{1}{2}}{2} = 1/4$

$\Rightarrow y(x) = x^{1/2} Z_{1/4}(x^2) = x^{1/2}\left[C_1 J_{1/4}(x^2) + C_2 N_{1/4}(x^2) \right]$

⑥ problem 12.16.14! solve $x^2y'' + xy' + (4x^2 - 9)y = 0$

we can simply solve it as we have solved the previous problem. or if you want we can get the solution directly by comparing the equation with equation (16.5) $x^2y'' + xy' + (K^2x^2 - p^2)y = 0$, with solution

$y(x) = C_1 J_p(Kx) + C_2 N_p(Kx)$; with $K = 2$ and $p = 3$

$= C_1 J_3(2x) + C_2 N_3(2x)$

⑦ problem 12.17.3! given that

$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$, Find $J_{3/2}(x)$ and $J_{-3/2}(x)$.

using the recursion relation 15.3 $J_{p-1} + J_{p+1} = \frac{2p}{x} J_p$

let $p = 1/2 \Rightarrow J_{-1/2} + J_{3/2} = \frac{1}{x} J_{1/2} \Rightarrow J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2}$

$$\Rightarrow J_{3/2} = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

Now let $p = -1/2 \Rightarrow J_{-3/2} + J_{1/2} = -\frac{1}{x} J_{-1/2}$

$$\Rightarrow J_{-3/2} = -\frac{1}{x} J_{-1/2} - J_{1/2}$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

similarly by taking $p = 3/2$, $J_{1/2} + J_{5/2} = \frac{3}{x} J_{3/2}$

$$\Rightarrow J_{5/2} = \frac{3}{x} J_{3/2} - J_{1/2} = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right]$$

and by taking $p = -3/2 \Rightarrow J_{-5/2} + J_{-1/2} = -\frac{3}{x} J_{-3/2}$

$$\Rightarrow J_{-5/2} = \frac{-3}{x} J_{-3/2} - J_{-1/2} = \frac{-3}{x}\left(-\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)\right) - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right]$$

⑧ problem 12.17.3: Find $N_{1/2}(x)$, $N_{3/2}(x)$, $N_{5/2}(x)$

using the relation $N_{n+1/2}(x) = (-1)^{n+1} J_{-(n+1/2)}(x)$

take $n=0 \Rightarrow N_{1/2}(x) = -J_{-1/2} = -\sqrt{\frac{2}{\pi x}} \cos x$

$n=1 \Rightarrow N_{3/2}(x) = -J_{-3/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{\cos x}{x} + \sin x\right]$

$n=2 \Rightarrow N_{5/2}(x) = -J_{-5/2} = -\sqrt{\frac{2}{\pi x}} \left[\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right]$

now taking negative values yields

$n=-1 \Rightarrow N_{-1/2}(x) = (-1)^0 J_{1/2} = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$n=-2 \Rightarrow N_{-3/2}(x) = (-1)^{-1} J_{3/2} = \frac{1}{(-1)^1} = -J_{3/2}(x)$

$= -\sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x\right]$

$n=-3 \Rightarrow N_{-5/2}(x) = (-1)^{-2} J_{5/2}(x) = \frac{1}{(-1)^2} J_{5/2} = J_{5/2}(x)$

$= \sqrt{\frac{2}{\pi x}} \left\{\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right\}$

see that the above results can be verified using the basic relation of N_p

$$N_p(x) = \frac{\cos \pi p \, J_p(x) - J_{-p}(x)}{\sin \pi p}$$

; just set $p = \pm 1/2, \pm 3/2, \pm 5/2$

(9) problem 12.17.4:

show that $I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$ and $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$

$I_p(x) = i^{-p} J_p(+ix) \Rightarrow I_{1/2}(x) = i^{-1/2} J_{1/2}(ix)$

using $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \Rightarrow I_{1/2}(x) = +i^{-1/2}\left[\sqrt{\frac{2}{\pi i x}} \sin ix\right]$

$\Rightarrow I_{1/2}(x) = \frac{1}{\sqrt{i}} \frac{1}{\sqrt{i}} \left[\sqrt{\frac{2}{\pi x}} \sin ix\right]$; using $\sin ix = i \sinh x$

$= \frac{1}{\cancel{i}}\left[\sqrt{\frac{2}{\pi x}} \cancel{i} \sinh x\right] = \sqrt{\frac{2}{\pi x}} \sinh x$

Now $K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix) \Rightarrow K_{1/2}(x) = \frac{\pi}{2} i^{3/2} H_{1/2}^{(1)}(ix)$

using $H_p^{(1)}(x) = J_p + iN_p \Rightarrow H_{1/2}^{(1)} = J_{1/2} + iN_{1/2}$

$\Rightarrow K_{1/2}(x) = \frac{\pi}{2} i^{3/2}\left\{J_{1/2}(ix) + iN_{1/2}(ix)\right\}$

$= \frac{\pi}{2} i^{3/2}\left[\sqrt{\frac{2}{\pi i x}} \sin ix + i\left(-\sqrt{\frac{2}{\pi i x}} \cos ix\right)\right]$

$= \frac{\pi}{2} i \left[\sqrt{\frac{2}{\pi x}} \sin ix - i\sqrt{\frac{2}{\pi x}} \cos ix\right]$

Now using $\sin ix = i \sinh x$ and $\cos ix = \cosh x$

$\Rightarrow K_{1/2}(x) = \frac{\pi}{2} i \left[\sqrt{\frac{2}{\pi x}}\, i \sinh x - i\sqrt{\frac{2}{\pi x}} \cosh x\right]$

$= -\frac{\pi}{2}\sqrt{\frac{2}{\pi x}}\left[\sinh x - \cosh x\right] = \sqrt{\frac{\pi}{2x}}\left\{\sinh x - \cosh x\right\}$

$= \sqrt{\frac{\pi}{2x}}\left\{\frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2}\right\} = \sqrt{\frac{\pi}{2x}} e^{-x}$

⑩ problem 12.19.2! given that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$

evaluate $\int_0^1 \left(\frac{\sin \alpha x}{\alpha x} - \cos \alpha x\right)^2 dx$

Now $J_{3/2}(\alpha x) = \sqrt{\frac{2}{\pi \alpha x}}\left(\frac{\sin \alpha x}{\alpha x} - \cos \alpha x\right) \Rightarrow$

$$\Rightarrow \left(\frac{\sin \alpha x}{\alpha x} - \cos \alpha x\right) = \sqrt{\frac{\pi \alpha x}{2}}\, J_{3/2}(\alpha x)$$

$$\Rightarrow \int_0^1 \left(\frac{\sin \alpha x}{\alpha x} - \cos \alpha x\right)^2 dx = \int_0^1 \left(\sqrt{\frac{\pi \alpha x}{2}}\, J_{3/2}(\alpha x)\right)^2 dx$$

$$= \frac{\pi \alpha}{2}\int_0^1 x\, J_{3/2}^2(\alpha x)\, dx = \frac{\pi \alpha}{2}\int_0^1 x\, J_{3/2}(\alpha x)\, J_{3/2}(\alpha x)\, dx$$

$$= \frac{\pi \alpha}{2} \cdot \frac{1}{2} J_{\frac{1}{2}}^2(\alpha) = \frac{\pi \alpha}{2} \cdot \frac{1}{2} \cdot \left[\frac{2}{\pi \alpha} \sin^2 \alpha\right] = \frac{1}{2}\sin^2 \alpha$$

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⑪ problem 12.23.14 show that $\int x^3 J_0(x)\, dx = x^3 J_1 - 2x^2 J_2$

$\int x^3 J_0\, dx = \int x^2 x^1 J_0\, dx$; using $\int x^p J_{p-1}\, dx = x^p J_p$

integrate by parts

let $u = x^2$, $dv = x^1 J_0\, dx$

$du = 2x\, dx$, $v = \int x^1 J_0\, dx = x J_1$

$$\Rightarrow \int x^3 J_0\, dx = x^2 (x J_1) - 2\underbrace{\int x^2 J_1\, dx}$$

$$= x^3 J_1 - 2x^2 J_2 + \text{constant}$$