Mathematical Physics (2)
HW # 6 – Solution
Dr. Gassem Alzoubi

(1) show that by the ratio test that $J_p(x)$ series is convergent for all x values. $\boxed{\text{problem 12.12(a)}}$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \frac{x^{2n+p}}{2^{2n+p}} \quad ; \text{ but } n! = \Gamma(n+1)$$

$$= \frac{x^p}{2^p} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n}}{2^{2n}} = \frac{x^p}{2^p} \sum_{n=0}^{\infty} a_{2n}\, x^{2n} \quad ; \text{ where}$$

$$a_{2n} = \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{1}{2^{2n}} \quad ; \text{ so, } a_{2(n+1)} = \frac{(-1)^{n+1}}{\Gamma(n+2)\Gamma(n+p+2)\, 2^{2n+2}}$$

$$= -\frac{(-1)^n}{\Gamma(n+2)\Gamma(n+p+2)\, 2^{2n+2}}$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{2(n+1)}}{a_{2n}} \right| = \lim_{n\to\infty} - \frac{\dfrac{(-1)^n}{\Gamma(n+2)\Gamma(n+p+2)\, 2^{2n+2}}}{\dfrac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)\, 2^{2n}}}$$

$$= \lim_{n\to\infty} - \frac{\Gamma(n+1)\Gamma(n+p+1)}{\Gamma(n+2)\Gamma(n+p+2)\cdot 4} \quad ; \quad \text{using } \Gamma(n+2) = (n+1)\Gamma(n+1) \; ; \; \Gamma(n+p+2) = (n+p+1)\Gamma(n+p+1)$$

$$= \lim_{n\to\infty} - \frac{\cancel{\Gamma(n+1)}\,\cancel{\Gamma(n+p+1)}}{(n+1)\cancel{\Gamma(n+1)}\cdot(n+p+1)\cancel{\Gamma(n+p+1)}\cdot 4} = -\frac{1}{4}\lim_{n\to\infty}\frac{1}{(n+1)(n+p+1)}$$

$$= 0$$

since $\lim \left| \dfrac{a_{2(n+1)}}{a_{2n}} \right| < 1 \Rightarrow$ the series is convergent

② problem 12.12.2: show that $\frac{2}{x}J_1(x) - J_0(x) = J_2(x)$

using $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+1)} \frac{x^{2n+p}}{2^{2n+p}}$, we have

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1)} \frac{x^{2n}}{2^{2n}} \;;\; J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n+1}}{2^{2n+1}}$$

$$\frac{2}{x}J_1 - J_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n}}{2^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1)} \frac{x^{2n}}{2^{2n}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}} \left[\frac{1}{\Gamma(n+1)\,\Gamma(n+2)} - \frac{1}{\Gamma(n+1)\,\Gamma(n+1)} \right]$$

$$= \sum_{n=0} \frac{(-1)^n x^{2n}}{2^{2n}\,\Gamma(n+1)} \left[\frac{1}{\Gamma(n+2)} - \frac{1}{\Gamma(n+1)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\,\Gamma(n+1)} \left[\frac{\Gamma(n+1) - \Gamma(n+2)}{\Gamma(n+2)\,\Gamma(n+1)} \right] \;;$$ using $\Gamma(n+2) = (n+1)\,\Gamma(n+1)$

$$= \sum_{n=0} \frac{(-1)^n x^{2n}}{2^{2n}\,\Gamma(n+1)} \left[\frac{\Gamma(n+1) - (n+1)\,\Gamma(n+1)}{\Gamma(n+2)\,\Gamma(n+1)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\,\Gamma(n+1)} \left[\frac{-n}{\Gamma(n+2)} \right] \;;$$ note that the $n=0$ term vanish, so the series effectivily starts at $n=1$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\,\Gamma(n+1)} \left[\frac{-n}{\Gamma(n+2)} \right]$$

(let us make the series start from $n=0$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2}\,\Gamma(n+2)} \left[\frac{-(n+1)}{\Gamma(n+3)} \right]$$

$$\Rightarrow \quad = \sum_{n=0}^{\infty} \frac{(-1)(-1)^n x^{2n+2}}{2^{2n+2}(n+1)\Gamma(n+1)} \left[-\frac{(n+1)}{\Gamma(n+3)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3)} \frac{x^{2n+2}}{2^{2n+2}} \equiv J_2(x) \quad \checkmark$$

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③ <u>problem 12.12.4</u> ; show that $\frac{d}{dx} J_0(x) = -J_1(x)$

starting from $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)} \frac{x^{2n}}{2^{2n}}$, we have

$$\frac{d}{dx} J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[\Gamma(n+1)]^2} \cdot 2n \frac{x^{2n-1}}{2^{2n}}$$; Note that the term $n=0$ vanish, so the series initially starts at $n=1$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{[\Gamma(n+1)]^2} \cdot 2n \frac{x^{2n-1}}{2^{2n}}$$; let us make the series starts at $n=0$

$$= \sum_{n=0} \frac{(-1)^{n+1}}{[\Gamma(n+2)]^2} \cdot 2(n+1) \frac{x^{2n+1}}{2^{2n+2}}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{[\Gamma(n+2)]^2} \cdot 2(n+1) \frac{x^{2n+1}}{2^{2n+2}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{[\Gamma(n+2)]^2} \frac{(n+1) x^{2n+1}}{2^{2n+1}}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+2)}{[\Gamma(n+2)]^2 \Gamma(n+1)} \frac{x^{2n+1}}{2^{2n+1}}$$; where I used $\Gamma(n+2) = (n+1)\Gamma(n+1)$ $\Rightarrow (n+1) = \frac{\Gamma(n+2)}{\Gamma(n+1)}$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)} \frac{x^{2n+1}}{2^{2n+1}}$$

$$= -J_1(x)$$

④ <u>problem 12.12.5:</u> show that $\frac{d}{dx} x J_1(x) = x J_0(x)$

starting from $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n+1}}{2^{2n+1}}$

$$\Rightarrow x J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n+2}}{2^{2n+1}}$$

$$\Rightarrow \frac{d}{dx} x J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} (2n+2) \frac{x^{2n+1}}{2^{2n+1}}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{2}(n+1)}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n}}{\cancel{2}\cdot 2^{2n}} = x \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(n+1)}\, x^{2n}}{\Gamma(n+1)\,\cancel{(n+1)}\,\Gamma(n+1)\, 2^{2n}}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1)} \frac{x^{2n}}{2^{2n}} = x J_0(x)$$

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⑤ <u>problem 12.12.7:</u> show that $\lim_{x\to 0} \frac{J_1(x)}{x} = 1/2$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n+1}}{2^{2n+1}} \Rightarrow \frac{J_1(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \frac{x^{2n}}{2^{2n+1}}$$

$$\Rightarrow \frac{J_1(x)}{x} = \frac{1}{\Gamma(1)\,\Gamma(2)\cdot 2} + (\quad)x^2 + (\quad)x^4 + (\quad)x^6 + \cdots$$

$$= \frac{1}{2} + (\quad)x^2 + (\quad)x^4 + (\quad)x^6 + \cdots$$

See that

$$\lim_{x\to 0} \frac{J_1(x)}{x} = 1/2 + 0 + 0 + 0 + \cdots = 1/2$$

⑥ problem 12.13.2:

show that $J_{-p}(x) = (-1)^p J_p(x)$ done on class

and show that $J_p(-x) = (-1)^p J_p(x)$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n+p}}{2^{2n+p}} \quad , \text{ let } x \to -x$$

$$\Rightarrow J_p(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{(-x)^{2n+p}}{2^{2n+p}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} (-1)^{2n+p} \frac{x^{2n+p}}{2^{2n+p}}$$

$$\to = \underbrace{(-1)^{2n}}_{1} (-1)^p = (-1)^p$$

$$= (-1)^p \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n+p}}{2^{2n+p}} = (-1)^p J_p(x)$$

---

⑦ problem 12.13.3: show that $\sqrt{\frac{\pi x}{2}} J_{-1/2}(x) = \cos x$

$$J_{-1/2}(x) = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1/2) \cdot 2^{2n-1/2}} x^{2n} = \frac{x^{-1/2}}{2^{-1/2}} \sum_{n=0} \frac{(-1)^n x^{2n}}{\Gamma(n+1)\Gamma(n+1/2)\, 2^{2n}}$$

$$= \sqrt{\frac{2}{x}} \left\{ \frac{1}{\Gamma(1)\Gamma(1/2)} - \frac{1}{\Gamma(2)\Gamma(3/2)} \frac{x^2}{2^2} + \frac{1}{\Gamma(3)\Gamma(5/2)} \frac{x^4}{2^4} - \cdots \right. \quad \Gamma(1/2) = \sqrt{\pi}$$

$$= \sqrt{\frac{2}{x}} \left\{ \frac{1}{\sqrt{\pi}} - \frac{1}{\frac{1}{2}\Gamma(1/2)\, 2^2} x^2 + \frac{1}{2 \times \frac{3}{2} \times \frac{1}{2} \Gamma(1/2)\, 2^4} x^4 - \cdots \right.$$

$\Gamma(3/2) = 1/2\, \Gamma(1/2)$

$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2)$

$= \frac{3}{2} \cdot \frac{1}{2} \Gamma(1/2)$

$$= \sqrt{\frac{2}{x}} \left\{ \frac{1}{\sqrt{\pi}} - \frac{1}{2\sqrt{\pi}} x^2 + \frac{1}{24\sqrt{\pi}} x^4 - \cdots \right.$$

$$= \sqrt{\frac{2}{\pi x}} \underbrace{\left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots \right\}}_{\cos x} = \sqrt{\frac{2}{\pi x}} \cos x \checkmark$$

(8) problem 12.13.6: show that $N_{n+1/2}(x) = (-1)^{n+1} J_{-(n+1/2)}(x)$

starting from $N_p(x) = \dfrac{\cos \pi p \; J_p(x) - J_{-p}(x)}{\sin \pi p}$, let $p = n + 1/2$

$$N_{n+1/2}(x) = \frac{\cos \pi(n+1/2)\; J_{(n+1/2)} - J_{-(n+1/2)}}{\sin \pi(n+1/2)} = \frac{\cos(\pi n + \frac{\pi}{2}) J_{n+1/2} - J_{-(n+1/2)}}{\sin(\pi n + \pi/2)}$$

using $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\Rightarrow \cos(\pi n + \pi/2) = \cos n\pi \underbrace{\cos \frac{\pi}{2}}_{\text{zero}} - \underbrace{\sin n\pi}_{\text{zero}} \sin \frac{\pi}{2} = 0$$

and using $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$$\Rightarrow \sin(n\pi + \frac{\pi}{2}) = \underbrace{\sin n\pi}_{0} \underbrace{\cos \frac{\pi}{2}}_{0} + \underbrace{\cos n\pi}_{(-1)^n} \underbrace{\sin \frac{\pi}{2}}_{1} = (-1)^n$$

$$\Rightarrow N_{n+1/2} = -\frac{J_{-(n+1/2)}}{(-1)^n} = \frac{J_{-(n+1/2)}}{(-1)^1 (-1)^n} = \frac{J_{-(n+1/2)}}{(-1)^{n+1}} \; ; \text{ multiply by } \frac{(-1)^{n+1}}{(-1)^{n+1}}$$

$$= \frac{(-1)^{n+1} J_{-(n+1/2)}}{(-1)^{2n+2}} = (-1)^{n+1} J_{-(n+1/2)} \; ; \text{ as } (-1)^{2n+2} = +1$$

so for $n = 0 \Rightarrow N_{1/2} = -J_{-1/2}$

for $n = 1 \Rightarrow N_{3/2} = +J_{-3/2}$

for $n = 2 \Rightarrow N_{5/2} = -J_{-5/2}$, and so on

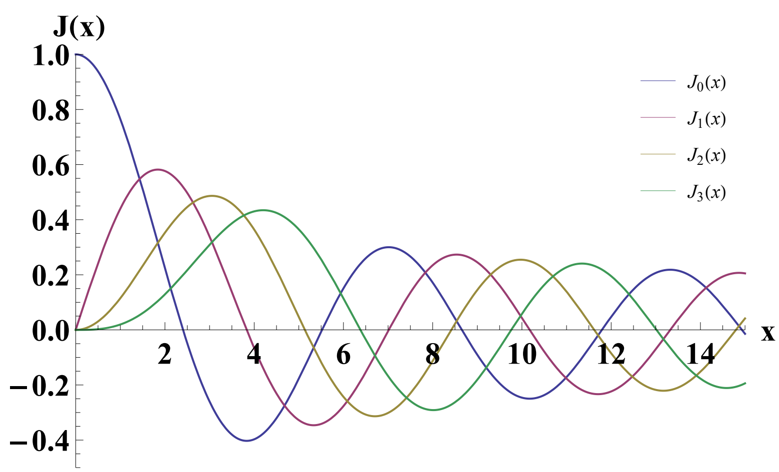
Problems of chapter 12 section 14

Bessel functions of first kind $J_p(x)$

Problem 12.14 .1 : By computer, plot graphs of $J_p(x)$ for p = 0, 1, 2, 3, and x from 0 to 15.

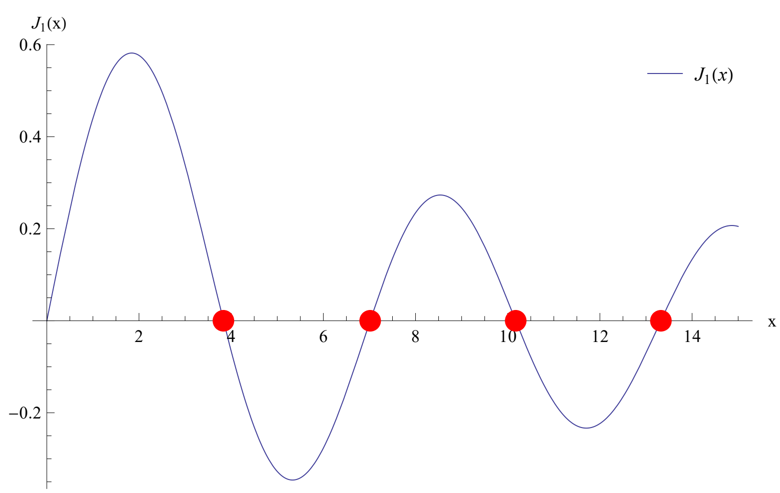
In[102]:=
```
MyBessel = Table[BesselJ[J, x], {J, 0, 3}];
Plot[MyBessel, {x, 0, 15}, ImageSize → 400, PlotRange → {{0, 15}, {-0.5, 1}},
 LabelStyle → {16, Bold}, PlotStyle → {Thickness[0.003]},
 AxesLabel → {"x", " J(x)"}, PlotLegends → Placed["Expressions", {Right, Top}]]
```

Out[102]=



In[101]:=
```
Plot[BesselJ[1, x], {x, 0, 15},
 Epilog → {PointSize[0.03], Red, Point[Table[{BesselJZero[1, k], 0}, {k, 4}]]},
 AxesLabel → {"x", " J1(x)"}, ImageSize → 400,
 PlotLegends → Placed["Expressions", {Right, Top}]]
```

Out[101]=



Problem 12.14 .2 : From the graphs in Problem 12.14 .1,
read approximate values of the first three zeros of each of the functions. Then,
by computer, find more accurate values of the zeros

```
N[BesselJZero[1, 3]]
```

```
10.1735
```

```
MyBesseljzeros = Table[BesselJZero[J, x], {J, 0, 3}];
```

```
MyBessel = Table[N[MyBesseljzeros], {x, 0, 3}];
```

```
Grid[MyBessel, Alignment → Left, Spacings → {2, 1}, Frame → All, ItemStyle → "Text"]
```

BesselJZero[0., 0.]	BesselJZero[1., 0.]	BesselJZero[2., 0.]	BesselJZero[3., 0.]
2.40483	3.83171	5.13562	6.38016
5.52008	7.01559	8.41724	9.76102
8.65373	10.1735	11.6198	13.0152

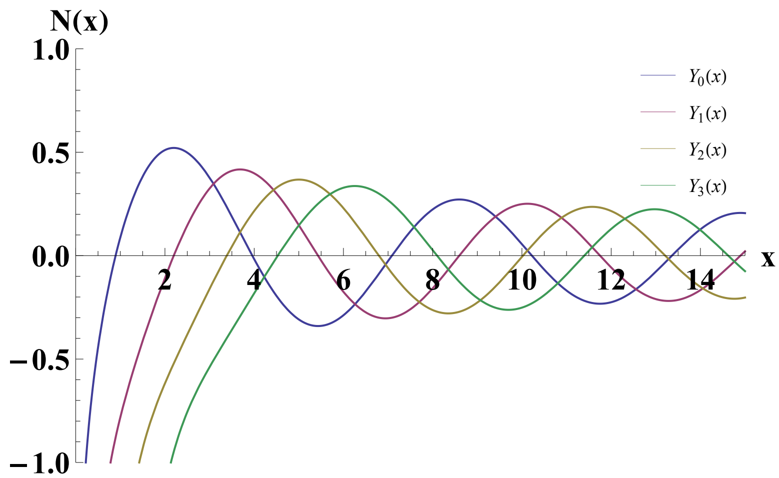
(∗Bessel functions of second kind $N_p(x)$∗)

Problem 12.14 .3 : By computer, plot N_p (x) for p = 0, 1, 2, 3, and x from 1 to 15

In[99]:=
```
MyBesselY = Table[BesselY[Y, x], {Y, 0, 3}];

Plot[MyBesselY, {x, 0, 15}, ImageSize → 400, PlotRange → {{0, 15}, {-1, 1}},
 LabelStyle → {16, Bold}, PlotStyle → {Thickness[0.003]},
 AxesLabel → {"x", " N(x)"}, PlotLegends → Placed["Expressions", {Right, Top}]]
```

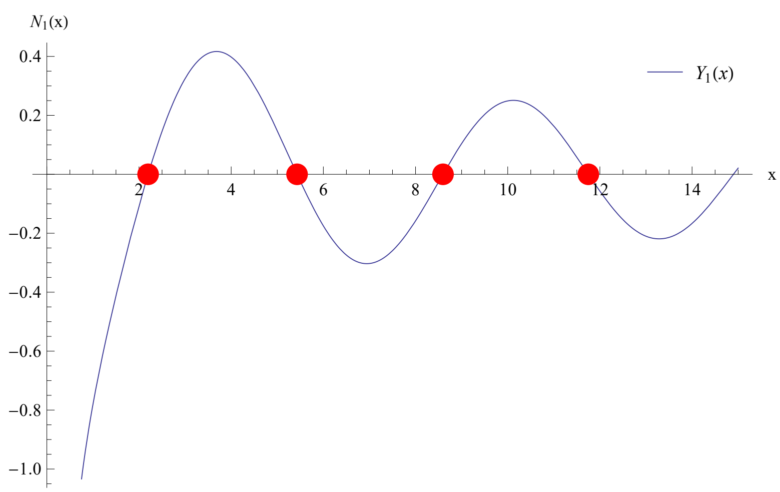
Out[100]=



In[103]:=

```
Plot[BesselY[1, x], {x, 0, 15},
 Epilog → {PointSize[0.03], Red, Point[Table[{BesselYZero[1, k], 0}, {k, 4}]]},
 AxesLabel → {"x", " N₁(x)"}, ImageSize → 400,
 PlotLegends → Placed["Expressions", {Right, Top}]]
```

Out[103]=



Problem 12.14 .4:
From the graphs in Problem 12.14 .3,
read approximate values of the first three zeros of each of the functions,
and then find more accurate values by computer

```
N[BesselYZero[1, 3]]
```

```
8.59601
```

```
MyBesselYzeros = Table[BesselYZero[N, x], {N, 0, 3}];
```

```
MyBesselY = Table[N[MyBesselYzeros], {x, 0, 3}];
```

```
Grid[MyBesselY, Alignment → Left, Spacings → {2, 1}, Frame → All, ItemStyle → "Text"]
```

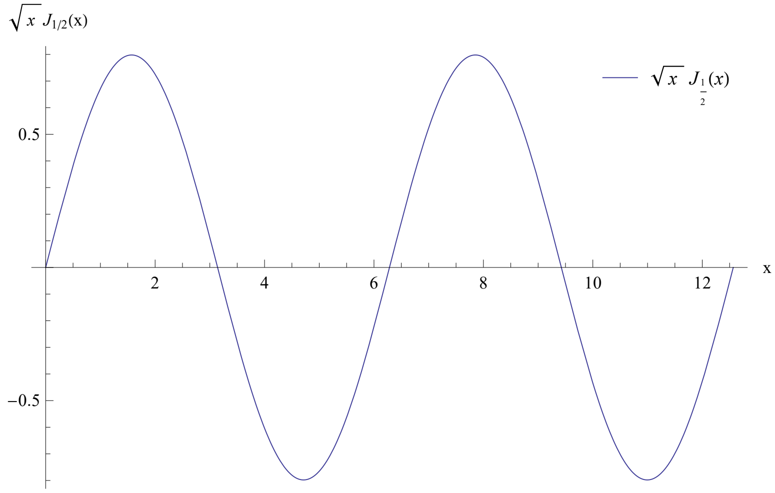
BesselYZero[0., 0.]	BesselYZero[1., 0.]	BesselYZero[2., 0.]	BesselYZero[3., 0.]
0.893577	2.19714	3.38424	4.52702
3.95768	5.42968	6.79381	8.09755
7.08605	8.59601	10.0235	11.3965

Problem 12.14 .5: By computer, plot $\sqrt{x}\, J_{1/2}(x)$ for x from 0 to 4π. Do you recognize the curve?

In[104]:=

```
Plot[√x * BesselJ[1 / 2, x], {x, 0, 4 Pi}, AxesLabel → {"x", " √x J1/2 (x) "},
 ImageSize → 400, PlotLegends → Placed["Expressions", {Right, Top}]]
```

Out[104]=



Problem 12.14 .6: By computer, find 30 zeros of J_0 (x) and note that the spacing between consecutive zeros is tending to π

```
MyBesseljzeros = Table[BesselJZero[J, x], {J, 0, 0}];
```

```
MyBessel = Table[N[MyBesseljzeros], {x, 0, 30}]
```

```
{{BesselJZero[0., 0.]}, {2.40483}, {5.52008}, {8.65373}, {11.7915},
 {14.9309}, {18.0711}, {21.2116}, {24.3525}, {27.4935}, {30.6346},
 {33.7758}, {36.9171}, {40.0584}, {43.1998}, {46.3412}, {49.4826},
 {52.6241}, {55.7655}, {58.907}, {62.0485}, {65.19}, {68.3315}, {71.473},
 {74.6145}, {77.756}, {80.8976}, {84.0391}, {87.1806}, {90.3222}, {93.4637}}
```