Mathematical Physics (2)
HW #5 - solution
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① **problem 12.7.4:** show that $\int_{-1}^{1} P_l(x)\, P'_{l-1}(x)\, dx = 0$

using equation 7-6 in text book

$$\int_{-1}^{1} P_l(x) \cdot (\text{any polynomial of degree} < l)\, dx = 0$$

we get $\int_{-1}^{1} P_l(x) \underbrace{P'_{l-1}(x)}_{\text{Polynomial of degree } (l-2) < l} dx = 0$

similarly $\int_{-1}^{1} P'_l(x)\, P_{l+1}(x) = \int_{-1}^{1} P_{l+1}(x) \underbrace{P'_l(x)}_{\text{Polynomial of degree } (l-1) < l+1} = 0$, as

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② **problem 12.7.5:** show that $\int_{-1}^{1} P_l(x) = 0$, for $l > 0$

Two methods

a) $\int_{-1}^{1} P_l(x)\, dx = \int_{-1}^{1} P_l(x)\, P_0(x)\, dx = 0$

as $P_0(x)$ and $P_l(x)$ are orthogonal on $(-1,1)$ for $l > 0$

b) using Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell = \frac{1}{2^\ell \ell!}\frac{d}{dx}\left[\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]$$

$$\Rightarrow \int_{-1}^{1} P_\ell(x)\,dx = \frac{1}{2^\ell \ell!}\int_{-1}^{1}\frac{d}{\cancel{dx}}\left[\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]\cancel{dx}$$

$$= \frac{1}{2^\ell \ell!}\left[\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]_{-1}^{1} = \text{zero}$$

as after repeated differentiations, the term (x^2-1) will servive giving zero at $x=\pm 1$

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(3) <u>problem 12.7.6!</u> show that $P_1(x)$ and $[P_\ell(x)]^2$ are orthogonal on $(-1,1]$

$\int_{-1}^{1} P_1(x)\, P_\ell^2(x)\, dx = 0$, as $P_\ell^2(x)$ is a polynomial of degree (2ℓ), and $2\ell \neq 1$ for all possible values of $\ell = 0, 1, 2, 3, 4, ---$

for example

$\ell=0$, $\int_{-1}^{1} P_1 P_0^2\, dx = \int_{-1}^{1} x\cdot(1)\,dx = 0$

$\ell=1$, $\int_{-1}^{1} P_1 P_1^2\, dx = \int_{-1}^{1} x\cdot(x)^2 = \int_{-1}^{1} x^3 = 0$

$\ell=2$, $\int_{-1}^{1} P_1 P_2^2\, dx = \int_{-1}^{1} \underbrace{x\cdot\frac{1}{4}(3x^2-1)^2}_{\text{odd}} = 0$

and so on

(4) problem 12.8.1 find the normalized function of $\cos nx$ on $(0, \pi)$

let $f(x) = N \cos nx$ $\Rightarrow N^2 \underbrace{\int_0^\pi \cos^2 nx \, dx}_{\pi/2} = 1$

$\Rightarrow N^2 \cdot \frac{\pi}{2} = 1 \Rightarrow N = \sqrt{\frac{2}{\pi}}$

$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \cos nx$

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(5) problem 12.8.2: Normalize $P_2(x)$ on $(-1, 1)$

let $f(x) = N P_2(x)$

$\Rightarrow \int_{-1}^{1} N^2 [P_2(x)]^2 dx = 1 \Rightarrow N^2 \int_{-1}^{1} \frac{1}{4}(3x^2 - 1)^2 dx = 1$

$\Rightarrow \frac{N^2}{4} \underbrace{\int_{-1}^{1} (9x^4 - 6x^2 + 1) dx}_{\frac{16}{10}} = 1 \Rightarrow \frac{N^2}{4} \cdot \frac{16}{10} = 1$

$\Rightarrow N^2 \cdot \frac{4}{10} = 1$

$N^2 \cdot \frac{2}{5} = 1 \Rightarrow$

$N = \sqrt{\frac{5}{2}}$

$\Rightarrow f(x) = \sqrt{\frac{5}{2}} P_2(x)$

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(6) problem 12.8.4: Normalize $e^{-x^2/2}$ on $(-\infty, \infty)$

let $f(x) = N e^{-x^2/2}$ $\Rightarrow N^2 \int_{-\infty}^{\infty} [e^{-x^2/2}]^2 dx = 1$

$N^2 \underbrace{\int_{-\infty}^{\infty} e^{-x^2} dx}_{\sqrt{\pi}} = 1$ $\Rightarrow N^2 \cdot \sqrt{\pi} = 1 \Rightarrow N^2 = \frac{1}{\pi^{1/2}}$

$\Rightarrow N = \left(\frac{1}{\pi}\right)^{1/4}$

$\Rightarrow f(x) = \left(\frac{1}{\pi}\right)^{1/4} e^{-x^2/2}$

⑦ expand the following function in Legendre series
(12.9.2)

$$f(x) = \begin{cases} 0, & -1<x<0 \\ x, & 0<x<1 \end{cases} \quad ; \quad f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \text{, where}$$

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx = \frac{2l+1}{2} \int_{0}^{1} x P_l(x) dx$$

$$l=0 \Rightarrow A_0 = \frac{1}{2} \int_0^1 x P_0(x) dx = \frac{1}{2} \int_0^1 x(1) dx = \frac{1}{2} \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{4}$$

$$l=1 \Rightarrow A_1 = \frac{3}{2} \int_0^1 x P_1(x) dx = \frac{3}{2} \int_0^1 x \cdot x \, dx = \frac{3}{2} \frac{x^3}{3} \Big]_0^1 = \frac{1}{2}$$

$$l=2 \Rightarrow A_2 = \frac{5}{2} \int_0^1 x P_2(x) dx = \frac{5}{2} \int_0^1 x \cdot \frac{1}{2}(3x^2-1) dx = \frac{5}{16}$$

and so on

$$\Rightarrow f(x) = \begin{cases} 0, & -1<x<0 \\ x, & 0<x<1 \end{cases} = \frac{1}{4} P_0 + \frac{1}{2} P_1 + \frac{5}{16} P_2 - - - - -$$

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⑧ problem 12.9.12: expand $x - x^3$ in Legendre series

$$f(x) = x - x^3 \Rightarrow f(x) = \sum_{l=0}^{\infty} A_l P_l(x) = A_0 P_0 + A_1 P_1 + A_2 P_2 + A_3 P_3 + \cdots$$

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx = \frac{2l+1}{2} \int_{-1}^{1} (x - x^3) P_l(x) dx$$

$$\Rightarrow l=0 \Rightarrow A_0 = \frac{1}{2} \int_{-1}^{1} (x-x^3) P_0(x) dx = \frac{1}{2} \int_{-1}^{1} (x-x^3)(1) dx = 0$$

$$\Rightarrow l=1, \ A_1 = \frac{3}{2} \int_{-1}^{1} (x-x^3) P_1(x) dx = \frac{3}{2} \int_{-1}^{1} (x-x^3)(x) dx = \frac{2}{5}$$

$$l=2, \ A_2 = \frac{5}{2} \int_{-1}^{1} (x-x^3) \cdot P_2(x) dx = \frac{5}{2} \int_{-1}^{1} (x-x^3) \cdot \frac{1}{2}(3x^2-1) dx = 0$$

$$l=3, \ A_3 = \frac{7}{2} \int_{-1}^{1} (x-x^3) P_3(x) dx = \frac{7}{2} \int_{-1}^{1} (x-x^3) \cdot \frac{1}{2}(5x^3 - 3x) dx = -\frac{2}{5}$$

all the rest coefficients are zeros i.e $A_4 = A_5 = A_6 = \cdots = \text{zero}$

$$\Rightarrow f(x) = x - x^3 = \frac{2}{5} P_1 - \frac{2}{5} P_3 = \frac{2}{5}(P_1 - P_3)$$

<u>problem</u>: show that $\delta(1-x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x)$

$\therefore f(x) = \delta(1-x)$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$, $a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$

$\Rightarrow a_n = \frac{2n+1}{2} \int_{-1}^{1} \delta(1-x) P_n(x) dx$

$$a_0 = \frac{1}{2} \int_{-\infty}^{\infty} \delta(x-1) P_0(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \delta(x-1)\, dx = 1/2$$

$$a_1 = \frac{3}{2} \int_{-\infty}^{\infty} \delta(x-1) P_1(x)\, dx = \frac{3}{2} \int_{-\infty}^{\infty} \delta(x-1) \cdot x\, dx = 3/2$$

$$a_2 = \frac{5}{2} \int_{-\infty}^{\infty} \delta(x-1) P_2(x)\, dx = \frac{5}{2} \int_{-\infty}^{\infty} \delta(x-1) \cdot \frac{1}{2}(3x^2-1)\, dx$$

$$= \frac{5}{2} \cdot \frac{1}{2}(3-1) = 5/2$$

$a_3 = 7/2$, and so on $\Rightarrow a_n = \frac{2n+1}{2}$; $n = 0, 1, 2, \dots$

$$\Rightarrow \delta(1-x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) \checkmark$$

i.e

$$= a_0 P_0 + a_1 P_1 + a_2 P_2 + \cdots$$

$$= \frac{1}{2} P_0 + \frac{3}{2} P_1 + \frac{5}{2} P_2 + \frac{7}{2} P_3 + \cdots$$

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similarly one can show that

$$\delta(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{2} P_n(x)$$

problem: let $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ --- (1)

show that $\int_{-1}^{1} \{f(x)\}^2 dx = \sum_{n=0}^{\infty} \frac{2}{2n+1} a_n^2$

squaring both sides of (1) and integrating $\int_{-1}^{1}$, we get

$$\int_{-1}^{1} \{f(x)\}^2 dx = \int_{-1}^{1} \left[\sum_n a_n P_n(x) \sum_m a_m P_m(x)\right] dx$$

$$= \sum_n \sum_m \int_{-1}^{1} a_n a_m P_n(x) P_m(x) dx$$

$$= \sum_n a_n \sum_m a_m \int_{-1}^{1} P_n(x) P_m(x) dx$$

$$= \sum_n a_n \sum_m a_m \frac{2}{2m+1} \delta_{mn}$$

$$= \sum_n a_n \cdot \frac{2 a_n}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{2 a_n^2}{2n+1}$$

(9) problem 12.10.2: starting from

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]y=0,$$

make the change of variables $x=\cos\theta$ and obtain the associated Legendre equation

$x=\cos\theta \Rightarrow dx=-\sin\theta\, d\theta \Rightarrow \frac{d}{dx}=-\frac{1}{\sin\theta}\frac{d}{d\theta} \Rightarrow \frac{d}{d\theta}=-\sin\theta\frac{d}{dx}$

and $x^2=\cos^2\theta$, $1-x^2=1-\cos^2\theta=\sin^2\theta$

$\Rightarrow$ substitute into the above equation to get

$$\frac{1}{\cancel{\sin}\theta}(-\cancel{\sin}\theta)\frac{d}{dx}\left(\sin\theta(-\sin\theta)\frac{dy}{dx}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]=0$$

$$\frac{d}{dx}\left[\sin^2\theta\frac{dy}{dx}\right]+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]y=0,$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0$$

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0$$

$$(1-x^2)y''-2xy'+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0 \checkmark$$

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(10) problems 12.10.4 and 12.10.6

$P_1^1(x)=+(1-x^2)^{1/2}=+(1-\cos^2\theta)^{1/2}=+(\sin^2\theta)^{1/2}=+\sin\theta$

$\Rightarrow P_1^1(\cos\theta)=+\sin\theta$

and $P_3^2(x)=15x(1-x^2)=15\cos\theta(1-\cos^2\theta)=15\cos\theta\sin^2\theta$

$\Rightarrow P_3^2(\cos\theta)=15\cos\theta\sin^2\theta$

(11) problem evaluate $\int_{-1}^{1} P_2^2(x)\, P_2^2(x)\, dx$;

using $P_2^2(x) = 3(1-x^2)$

$$\Rightarrow \int_{-1}^{1} [P_2^2(x)]^2 dx = \int_{-1}^{1} 9(1-x^2)^2 = \int_{-1}^{1} 9(1+x^4-2x^2)$$

$$= 9\left[x + \frac{x^5}{5} - \frac{2}{3}x^3\right]_{-1}^{1} = \frac{48}{5}$$

or using $\int_{-1}^{1} P_{l'}^m(x)\, P_l^m(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$

$$\Rightarrow \int_{-1}^{1} P_2^2(x)\, P_2^2(x)\, dx = \frac{2}{5} \frac{4!}{0!} \delta_{22} \;;\; \delta_{22} = 1$$

$$= \frac{2}{5} \times 4\times3\times2 = \frac{48}{5}$$

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(12) problem evaluate $\int_{-1}^{1} P_1^1(x)\, P_2^1(x)\, dx$

using $P_1^1(x) = -(1-x^2)^{1/2}$ and $P_2^1(x) = -3x(1-x^2)^{1/2}$

$$\Rightarrow \int_{-1}^{1} P_1^1(x)\, P_2^1(x)\, dx = \int_{-1}^{1} -(1-x^2)^{1/2} . (-3x)(1-x^2)^{1/2} dx$$

$$= -\frac{3}{2} \int_{-1}^{1} -2x(1-x^2)^{1/2} dx = -\frac{3}{2} \frac{(1-x^2)^{3/2}}{3/2}$$

$$= -(1-x^2)^{3/2}\Big|_{-1}^{1} = 0$$

or using orthogonality

$$\int_{-1}^{1} P_1^1(x)\, P_2^1(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'} \;;$$

$l=2$, $l'=1$
$\delta_{ll'} = \delta_{21} = 0$
as $l \neq l'$

$$= 0$$