Mathematical Physics (2)
HW #4: solution
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① problem 12.2.1: Find $P_4(x)$

$l=4 \Rightarrow$ even series terminates $y_{even} = a_0 + a_2x^2 + a_4x^4$

$$a_2 = \frac{a_0(-l)(l+1)}{2.1} = a_0\frac{(-4)(5)}{2.1} = -10a_0$$

$$a_4 = \frac{a_2(2-l)(3+l)}{4.3} = \frac{(-10a_0)(-2)(7)}{(4)(3)} = \frac{35}{3}a_0$$

$$\Rightarrow y_{even} = a_0 - 10a_0x^2 + \frac{35}{3}a_0x^4 \; ; \quad y(1) = 1$$

$$1 = a_0\left(1 - 10 + \frac{35}{3}\right) = a_0\left(-9 + \frac{35}{3}\right) = a_0\left(-\frac{27}{3} + \frac{35}{3}\right)$$

$$= a_0\frac{8}{3} \quad \Rightarrow \quad a_0 = \frac{3}{8}$$

$$\Rightarrow y_{even}(x) = \frac{3}{8} - \frac{30}{8}x^2 + \frac{35}{8}x^4 \equiv P_4(x)$$

$$\therefore \boxed{P_4(x) = \frac{1}{8}(3 - 30x^2 + 35x^4)}$$

② Find the even series for $l=1$ and show that it diverges at $x=1$

$$y_{even} = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots$$

Now need to find $a_2, a_4, a_6, \cdots$

$$a_{n+2} = \frac{a_n(n-l)[n+l+1]}{(n+2)(n+1)} \; ; \quad \text{set } l=1$$

$$a_{n+2} = \frac{a_n(n-1)\cancel{[n+2]}}{\cancel{(n+2)}(n+1)} = a_n\frac{n-1}{n+1}$$

$\therefore \quad a_{n+2} = a_n \dfrac{n-1}{n+1}$

$n=0, \quad a_2 = -a_0$

$n=2, \quad a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0$

$n=4, \quad a_6 = \frac{3}{5} a_4 = \frac{3}{5}\left(-\frac{1}{3} a_0\right) = -\frac{1}{5} a_0$

$n=6, \quad a_8 = \frac{5}{7} a_6 = \frac{5}{7}\left(-\frac{1}{5} a_0\right) = -\frac{1}{7} a_0, \quad --$

$\Rightarrow y_{even} = a_0 - a_0 x^2 - \frac{1}{3} a_0 x^4 - \frac{1}{5} a_0 x^6 - \frac{1}{7} a_0 x^8 ----$

$= a_0 \left(1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \frac{1}{7} x^8 ---\right)$

$y(1) = a_0 \left(\cancel{1} - \cancel{1} - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} ---\right)$

$= a_0 \left(-\frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} ---- \frac{1}{2n+1} ----\right)$

$\Rightarrow$ using integral test

$$\int_1^\infty \frac{-a_0}{2n+1}\, dn = -a_0 \int_1^\infty \frac{1}{2n+1}\, dn = -\frac{a_0}{2} \int_1^\infty \frac{2}{2n+1}\, dn$$

$$= -\frac{a_0}{2} \ln(2n+1)\Big|_1^\infty \longrightarrow -\infty$$

diverges

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③ problem 12.2.2: show that $P_\ell(-x) = (-1)^\ell P_\ell(x)$

using $P_\ell(x) = \dfrac{1}{2^\ell \ell!} \dfrac{d^\ell}{dx^\ell} (x^2-1)^\ell$; set $x \to -x$

$$\Rightarrow P_\ell(-x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{d(-x)}\right)^\ell \left((-x)^2 - 1\right)^\ell = \frac{1}{2^\ell \ell!} \left(-\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

$$= (-1)^\ell \underbrace{\frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell}_{P_\ell(x)} = (-1)^\ell P_\ell(x)$$

④ problem (12.2.3)

In[208]:=

Computer plot graphs of $P_l(x)$ for $l = 0, 1, 2, 3, 4$, and x from −1 to 1.

use mathematica

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MyLegendre = Table[LegendreP[l, x], {l, 0, 4}];
Plot[MyLegendre, {x, -1, 1}, ImageSize → 800, AxesLabel → {"x", " Pl(x)"},
 PlotRange → {{-1, 1}, {-1, 1}}, LabelStyle → {16, Bold}, PlotStyle → {Thickness[0.003]}]
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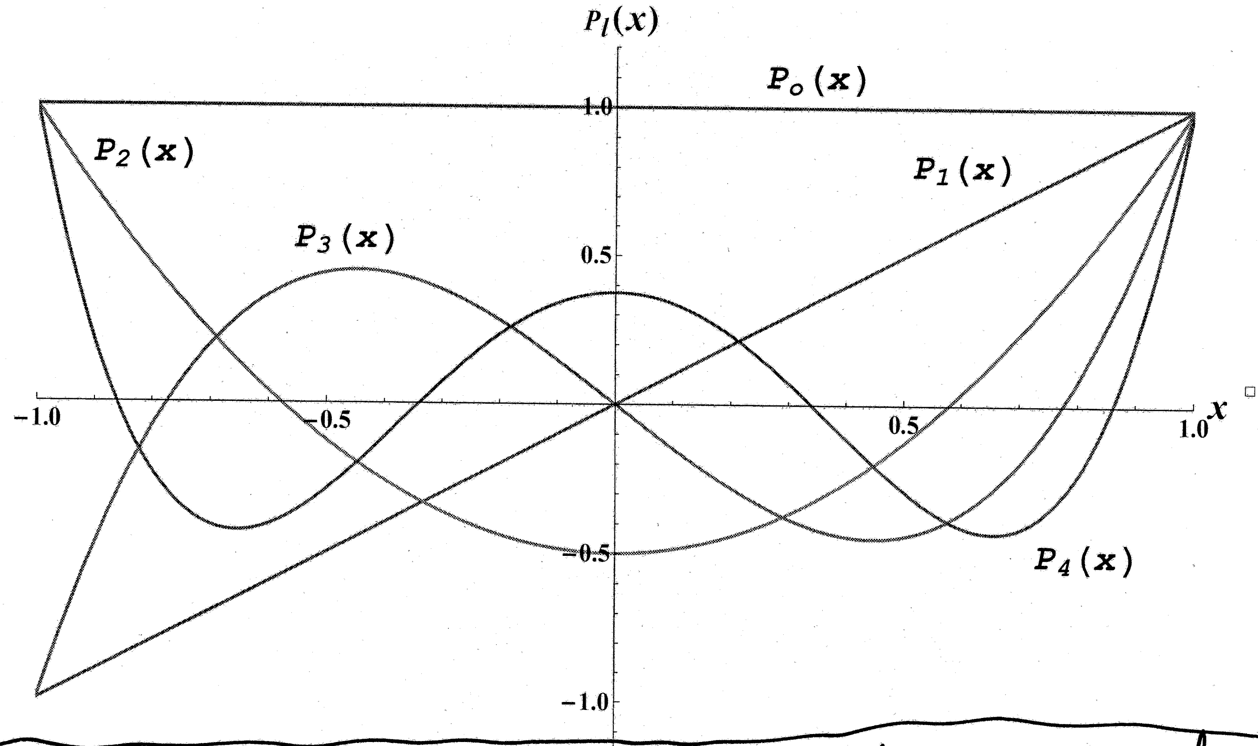

⑤ problem 12.4.3: using Rodrigues' formula, Find P_0, P_1, P_2, P_3

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l$$

① $P_0 = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2-1)^0 = 1$

② $P_1 = \frac{1}{2^1 1!} \frac{d^1}{dx^1} (x^2-1)^1 = \frac{1}{2}(2x) = x$

③ $P_2 = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{4\times 2} \frac{d}{dx} [2(x^2-1)(2x)] = \frac{1}{2} \frac{d}{dx} (x^3-x)$
$= \frac{1}{2}(3x^2-1)$

④ $P_3 = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3 = \frac{1}{8\times 6} \frac{d^2}{dx^2} [3(x^2-1)^2(2x)] = \frac{1}{8} \frac{d^2}{dx^2} [x^5+x-2x^3]$
$= \frac{1}{8} \frac{d}{dx} [5x^4+1-6x^2] = \frac{1}{8} [20x^3-12x] = \frac{5}{2}x^3 - \frac{3}{2}x$
$= \frac{1}{2}(5x^3-3x)$

⑤ $P_4 =$ ---- do it by yourself

⑥ problem 12.5.3 use the recursion relation

$l P_l(x) = (2l-1)\, x P_{l-1}(x) - (l-1) P_{l-2}(x)$ to find $P_2, P_3, P_4, \ldots$ given that $P_0 = 1,\ P_1 = x$

- $l=2 \Rightarrow 2P_2 = 3xP_1 - P_0 = 3x^2 - 1$

$\Rightarrow P_2 = \frac{1}{2}(3x^2-1)$ ✓

- $l=3 \Rightarrow 3P_3 = 5xP_2 - 2P_1 = 5x \cdot \frac{1}{2}(3x^2-1) - 2x$

$= \frac{15}{2}x^3 - \frac{9}{2}x \Rightarrow P_3 = \frac{5}{2}x^3 - \frac{3}{2}x = \frac{1}{2}(5x^3-3x)$ ✓

- $l=4 \Rightarrow 4P_4 = 7xP_3 - 3P_2 = 7 \cdot \frac{1}{2}(5x^3-3x) - \frac{3}{2}\cdot(3x^2-1)$

$= \frac{35}{2}x^4 - \frac{30}{2}x^2 + \frac{3}{2} \Rightarrow P_4 = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$

$= \frac{1}{8}(35x^4 - 30x^2 + 3)$ ✓

and so on

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⑦ problem 12.5.4: starting from the generating function $\Phi(x,h)$, show that $(x-h)\frac{\partial\Phi}{\partial x} = h\frac{\partial\Phi}{\partial h}$.

$$\Phi(x,h) = (1-2xh+h^2)^{-1/2} = \sum_{l=0}^{\infty} h^l P_l(x) \quad \text{---(1)}$$

$$\frac{\partial\Phi}{\partial h} = -\tfrac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = (1-2xh+h^2)^{-3/2}(x-h) \quad \text{---(2)}$$

$$\frac{\partial\Phi}{\partial x} = -\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2h) = h(1-2xh+h^2)^{-3/2} \quad \text{---(3)}$$

substitute in (2)

$$\Rightarrow (1-2xh+h^2)^{-3/2} = \frac{1}{h}\frac{\partial\Phi}{\partial x}$$

$$\Rightarrow \frac{\partial\Phi}{\partial h} = \frac{1}{h}\frac{\partial\Phi}{\partial x}(x-h) \Rightarrow \boxed{(x-h)\frac{\partial\Phi}{\partial x} = h\frac{\partial\Phi}{\partial h}} \ ✓ \quad \text{---(4)}$$

now from (4) $(x-h)\frac{d}{dx}\sum_{l=0}^{\infty} h^l P_l(x) = h\frac{\partial}{\partial h}\sum_{l=0}^{\infty} h^l P_l(x)$

$$\Rightarrow (x-h)\sum_{l=0}^{\infty} h^l P_l'(x) = h\sum_{l=0}^{\infty} l h^{l-1} P_l(x)$$

$$\Rightarrow x\sum_0 h^l P_l'(x) - \sum_{l=0}^{\infty} h^{l+1} P_l'(x) = \sum_{l=0}^{\infty} l h^l P_l(x)$$

$$\sum_0 x h^l P_l' - \sum_{l=1} h^l P_{l-1}' = \sum_{l=1} l h^l P_l$$

$\hookrightarrow$ at $l=0$, $x P_0'(x) = x[1]' =$ zero, so safe to start from $l=1$

$$\sum_{l=1} x h^l P_l' - \sum_{l=1} h^l P_{l-1}' = \sum_{l=1} l h^l P_l$$, equate coefficients of h^l

$$x P_l'(x) - P_{l-1}'(x) = l P_l(x) \checkmark$$

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⑧ problem 12.5.11: Express $x - x^3$ as linear combinations of Legendre polynomials

$P_0 = 1$, $P_1 = x$

$P_2 = \frac{1}{2}(3x^2 - 1)$

$P_3 = \frac{1}{2}(5x^3 - 3x)$

from $P_3 = \frac{1}{2}(5x^3 - 3x)$

$$2P_3 = 5x^3 - 3x$$

$$\Rightarrow x^3 = \frac{2}{5}P_3 + \frac{3}{5}\underbrace{x}_{P_1}$$

$$= \frac{2}{5}P_3 + \frac{3}{5}P_1$$

$$\Rightarrow x - x^3 = P_1 - \left(\frac{2}{5}P_3 + \frac{3}{5}P_1\right) = P_1 - \frac{2}{5}P_3 - \frac{3}{5}P_1$$

$$= \frac{2}{5}P_1 - \frac{2}{5}P_3 = \frac{2}{5}(P_1 - P_3)$$

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⑨ problem 12.6.2: show that the functions $e^{\frac{in\pi x}{l}}$; $n = 0, \pm 1, \pm 2, \ldots$ are a set of orthogonal functions on the interval $(-l, l)$

Here, we need to prove that

$$\int_{-l}^{l} e^{-i\frac{n\pi x}{l}} e^{i\frac{m\pi x}{l}} dx = 0, \quad \text{for } m \neq n$$

$$\Rightarrow \int_{-l}^{l} e^{-i\frac{n\pi x}{l}} e^{i\frac{m\pi x}{l}} dx = \int_{-l}^{l} e^{i\frac{\pi x}{l}(m-n)} dx = \frac{1}{i\frac{\pi}{l}(m-n)} \left. e^{i\frac{\pi x}{l}(m-n)} \right|_{-l}^{l}$$

$$= \frac{l}{i\pi(m-n)} \left[e^{i\pi(m-n)} - e^{-i\pi(m-n)} \right]$$

$$= \frac{l}{i\pi(m-n)} \left[\cos(m-n)\pi + i \underbrace{\sin(m-n)\pi}_{\text{zero}} - \cos(m-n)\pi - i \underbrace{\sin(m-n)\pi}_{\text{zero}} \right]$$

$$= \frac{l}{i\pi(m-n)} \left[\cos(m-n)\pi - \cos(m-n)\pi \right] = \text{zero}$$

$\therefore$ the functions $e^{i\frac{n\pi x}{l}}$ form an orthogonal set on the interval $(-l, l)$

Note that for $m = n$, we have

$$\int_{-l}^{l} e^{i\frac{\pi x}{l}(m-n)} dx = \int_{-l}^{l} dx = 2l$$

$\Rightarrow$ in general

$$\int_{-l}^{l} e^{-i\frac{n\pi x}{l}} e^{i\frac{m\pi x}{l}} dx = \begin{Bmatrix} 0, & m \neq n \\ 2l, & m = n \end{Bmatrix} = 2l\,\delta_{mn}$$

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(10) problem 12.6.3 show that the functions $\{x^2, \sin x\}$ are orthogonal on $(-1, 1)$

$$\int_{-1}^{1} \underbrace{\underbrace{x^2}_{\text{even}} \underbrace{\sin x}_{\text{odd}}}_{\text{odd}} dx = \text{zero}$$

integrating an odd function over symmetric interval yields zero

(11) problem 12.6.5: show that $P_0(x)$ and $P_2(x)$ are orthogonal on $(-1,1)$

$$\int_{-1}^{1} P_0(x)\, P_2(x)\,dx = \int_{-1}^{1} (1)\cdot \tfrac{1}{2}(3x^2-1)\,dx = \tfrac{1}{2}\left[x^3-x\right]_{-1}^{1} = 0$$

where I used $P_0(x)=1$ and $P_2(x) = \frac{1}{2}(3x^2-1)$

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(12) problem 12.6.6: show that $P_\ell(x)$ and $P_\ell'(x)$ are orthogonal on $(-1,1)$

need to prove that $\int_{-1}^{1} P_\ell(x)\, P_\ell'(x)\,dx = 0$

integrate by parts

let $u = P_\ell(x)$, $dv = P_\ell'(x)\,dx$

$du = P_\ell'(x)\,dx$ $\quad v = P_\ell(x)$

$$\Rightarrow \int_{-1}^{1} P_\ell P_\ell'\,dx = \underbrace{P_\ell^2(x)\Big|_{-1}^{1}}_{\text{zero}} - \int_{-1}^{1} P_\ell P_\ell'\,dx \;; \qquad P_\ell(1)=1,\quad P_\ell(-1)=(-1)^\ell$$

$$= -\int_{-1}^{1} P_\ell P_\ell'\,dx$$

$$\Rightarrow 2\int_{-1}^{1} P_\ell P_\ell'\,dx = 0 \quad \Rightarrow \int_{-1}^{1} P_\ell P_\ell'\,dx = 0$$

$\Rightarrow P_\ell(x)$ and $P_\ell'(x)$ are orthogonal.

(13) problem 12.6.9: show that $\int_{-1}^{1} P_{2l+1}(x)\,dx = 0$

using $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$, we have for $P_{2l+1}(x)$

$$P_{2l+1}(x) = \frac{1}{2^{(2l+1)}(2l+1)!}\frac{d^{(2l+1)}}{dx^{(2l+1)}}(x^2-1)^{2l+1} = \frac{1}{2^{(2l+1)}(2l+1)!}\frac{d}{dx}\left(\frac{d^{2l}}{dx^{2l}}(x^2-1)^{2l+1}\right)$$

$$\Rightarrow \int_{-1}^{1} P_{2l+1}(x)\,dx = \frac{1}{2^{(2l+1)}(2l+1)!}\int_{-1}^{1}\frac{d}{\cancel{dx}}\left[\frac{d^{2l}}{dx^{2l}}(x^2-1)^{2l+1}\right]\cancel{dx}$$

$$= \frac{1}{2^{(2l+1)}(2l+1)!}\left[\frac{d^{2l}}{dx^{2l}}(x^2-1)^{2l+1}\right]_{-1}^{1}$$

= Zero,

because after repeated differentiations, the term (x^2-1) will servive problem: show that $\int_{-1}^{1} x P_l(x) P_{l-1}(x)\, dx = \frac{2l}{4l^2-1}$

using the identity (5.8 a),

$l P_l(x) = (2l-1) x P_{l-1}(x) - (l-1) P_{l-2}(x)$; multiply by ($P_l(x)$)

$l P_l^2 = (2l-1) x P_l P_{l-1} - (l-1) P_l P_{l-2}$; integrate both sides

$$l \int_{-1}^{1} P_l^2(x)\, dx = (2l-1) \int_{-1}^{1} x P_l P_{l-1}\, dx - (l-1) \underbrace{\int_{-1}^{1} P_l P_{l-2}\, dx}$$

zero as P_l, P_{l-2} are orthogonal

$$\Rightarrow \int_{-1}^{1} x P_l P_{l-1}\, dx = \frac{l}{2l-1} \int_{-1}^{1} P_l^2(x)\, dx$$

$$= \frac{l}{2l-1} \frac{2}{2l+1} = \frac{2l}{4l^2-1}$$

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problem: show that $\int_{-1}^{1} P_m(x) (1-2xh+h^2)^{-1/2} dx = \frac{2h^m}{2m+1}$

using $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$, we have

$$\Rightarrow \int_{-1}^{1} P_m(x) \sum_{n=0}^{\infty} h^n P_n(x)\, dx = \sum_{n=0}^{\infty} h^n \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

$$= \sum_{n=0}^{\infty} h^n \frac{2}{2m+1} \delta_{mn}$$

$$= \frac{2h^m}{2m+1}$$

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problem: show that $\frac{1}{\sqrt{2-2x}} = \sum_{l=0}^{\infty} P_l(x)$;

using $(1-2xh+h^2)^{-1/2} = \sum_{l=0}^{\infty} h^l P_l(x)$; put $h=1$

$(1-2x+1)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)$; where $(1)^l = 1$

$$\frac{1}{\sqrt{2-2x}} = \sum_{l=0}^{\infty} P_l(x)$$