Mathematical physics (2)
HW #3 - solution
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① problem 12.1.2: solve $y' = 3x^2 y$ by series method

$\Rightarrow y' - 3x^2 y = 0$, let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$\Rightarrow \sum_{n=0}^{\infty} n a_n x^{n-1} - 3x^2 \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_0 n a_n x^{n-1} - 3\sum_0 a_n x^{n+2} = 0$

let us unify powers to $(n+2)$ $\Rightarrow$ for the first sum replace n by $(n+3)$ inside the sum and decrease the sum index by 3 $\Rightarrow$

$\sum_{n=-3} (n+3) a_{n+3} x^{n+2} - 3\sum_0 a_n x^{n+2} = 0$; for the first sum, take out the terms with $n=-3,-2,-$

$$a_1 + 2a_2 x^1 + \sum_{n=0}^{\infty} (n+3) a_{n+3} x^{n+2} - 3\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x^1 + \sum_{n=0}^{\infty} \left[(n+3) a_{n+3} - 3a_n\right] x^n = 0 \equiv 0x^0 + 0x^1 + 0x^2 + \cdots$$

$\Rightarrow a_1 = 0$, $2a_2 = 0 \Rightarrow a_2 = 0$, and $(n+3)a_{n+3} - 3a_n = 0$

$\Rightarrow a_{n+3} = \frac{3a_n}{n+3}$; $n \geq 0$; $a_3 = a_0$, $a_4 = \frac{3}{4}a_1 = 0$, $a_5 = \frac{3}{5}a_2 = 0$

$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ $\quad$ $a_6 = \frac{3}{6}a_3 = \frac{1}{2}a_0$, $a_7 = \frac{3}{7}a_4 = 0$

$= a_0 + a_0 x^3 + \frac{a_0}{2}x^6 + \frac{a_0}{6}x^9 + \cdots$ $\quad$ $a_8 = 0$, $a_9 = \frac{1}{6}a_0$

$= a_0\left[1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \cdots\right] = a_0 e^{x^3}$,

$$= a_0 \underbrace{\left[1 + \frac{(x^3)^1}{1!} + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \frac{(x^3)^4}{4!} + \cdots\right]}_{= e^{x^3}}$$

where $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

we can verify the series solution by solving the differential equation using another technique; i.e

$y' = 3x^2 y \Rightarrow \frac{y'}{y} = 3x^2 \Rightarrow \frac{1}{y}\frac{dy}{dx} = 3x^2 \Rightarrow$

$\frac{dy}{y} = 3x^2 dx \Rightarrow$ integrate $\ln y = x^3 + \ln a_0$

$\Rightarrow \ln \frac{y}{a_0} = x^3 \Rightarrow y = a_0 e^{x^3}$ as expected

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(2) problem 12.1.3: solve $xy' = y$

$\Rightarrow xy' - y = 0$, let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$\Rightarrow x \sum_0 n a_n x^{n-1} - \sum_0 a_n x^n = 0 \Rightarrow \sum_0 n a_n x^n - \sum_0 a_n x^n = 0$

$\Rightarrow \sum_{n=0}^{\infty} a_n(n-1) x^n = 0 \Rightarrow a_n(n-1) = 0$

$n=0, \quad a_0(-1) = 0 \Rightarrow a_0 = 0$

$n=1, \quad a_1(0) = 0 \Rightarrow a_1 = \frac{0}{0}$ can't be determined

$n=3, \quad a_3(2) = 0 \Rightarrow a_3 = 0$

$n=4, \quad a_4(3) = 0 \Rightarrow a_4 = 0 = a_5 = a_6 = a_7 = 0 = \cdots$

$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$= a_1 x \quad \therefore \boxed{y = a_1 x}$

another method

$xy' = y \Rightarrow \frac{y'}{y} = \frac{1}{x}$, integrate $\ln y = \ln x + \ln a_1$

$= \ln a_1 x$

$\therefore \ln y = \ln a_1 x$

$\Rightarrow y = a_1 x$ as found above

③ problem 12.1.4: solve $y'' = -4y \Rightarrow y'' + 4y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{0}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{0} n(n-1) a_n x^{n-2}$

$\Rightarrow$ Substitute into the equation

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = 0, \Rightarrow$$

$$\sum_{n=-2} (n+2)(n+1) a_{n+2} x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 4a_n\right] x^n = 0, \quad a_{n+2} = \frac{-4a_n}{(n+2)(n+1)} \; ; n \geq 0$$

Here we have two arbitrary constants a_0, a_1

$a_2 = -2a_0$, $a_3 = -\frac{2}{3} a_1$, $a_4 = \frac{2}{3} a_0$, $a_5 = \frac{2}{15} a_1$, $a_6 = \frac{-4}{45} a_0$,

$a_7 = \frac{-4}{21 \times 15} a_1$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - 2a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{2}{3} a_0 x^4 + \frac{2}{15} a_1 x^5 - \frac{4}{45} a_0 x^6 - \frac{4}{21 \times 15} a_1 x^7 + \cdots$$

$$= a_0\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{21 \times 15}x^7 + \cdots\right)$$

$$= a_0\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right) +$$

$$a_1\left(x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right)$$

$$= a_0 \cos 2x + a_1 \sin 2x.$$

another method: $y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$
$= \alpha \pm \beta i$,
$\alpha = 0$, $\beta = 2$

$\Rightarrow y(x) = e^{\alpha x}(a_0 \cos \beta x + a_1 \sin \beta x)$

$= a_0 \cos 2x + a_1 \sin 2x$ as expected

④ problem 12.1.9: solve $(x^2+1)y'' - 2xy' + 2y = 0$

let $y=\sum_{n=0}^{\infty} a_n x^n$, $y'=\sum_{n=0}^{\infty} n a_n x^{n-1}$, $y''=\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$\Rightarrow (x^2+1)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_0 n(n-1)a_n x^n + \sum_0 n(n-1) x^{n-2} - 2\sum_0 n a_n x^n + 2\sum_0 a_n x^n = 0$

$\sum_0 n(n-1)a_n x^n + \sum_{n=-2} (n+2)(n+1)a_{n+2} x^n - 2\sum_0 n a_n x^n + 2\sum_0 a_n x^n = 0$

$\sum_0 n(n-1)a_n x^n + \sum_0 (n+2)(n+1)a_{n+2} x^n - 2\sum_0 n a_n x^n + 2\sum_0 a_n x^n = 0$

$\sum_0 [n(n-1)a_n + (n+2)(n+1)a_{n+2} - 2na_n + 2a_n] x^n = 0$

$\Rightarrow n(n-1)a_n + (n+2)(n+1)a_{n+2} - 2na_n + 2a_n = 0$

$a_n[n(n-1) - 2n + 2] + (n+2)(n+1)a_{n+2} = 0$

$\Rightarrow a_{n+2} = -\dfrac{[n(n-3)+2]a_n}{(n+2)(n+1)} \Rightarrow a_2 = -a_0,\ a_3 = 0,\ a_4 = 0 = a_5 = \cdots$

$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$= a_0 + a_1 x - a_0 x^2$

$= a_0(1-x^2) + a_1 x \quad \Rightarrow \boxed{y(x) = a_0(1-x^2) + a_1 x}$

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⑤ problem 12.1.5: solve $y'' = y$

$\Rightarrow y'' - y = 0$,

let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_0 n a_n x^{n-1}$, $y'' = \sum_0 n(n-1)a_n x^{n-2}$

$\Rightarrow \sum_0 n(n-1)a_n x^{n-2} - \sum_0 a_n x^n = 0$

$\sum_{-2} (n+2)(n+1)a_{n+2} x^n - \sum_0 a_n x^n = 0$

$$\Rightarrow \sum_{0} (n+2)(n+1)a_{n+2}x^n - \sum_{0} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n = 0 \Rightarrow (n+2)(n+1)a_{n+2} - a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

$$\Rightarrow n=0,\ a_2 = \frac{a_0}{2.1} = \frac{a_0}{2!}\ ;\ n=1,\ a_3 = \frac{a_1}{3.2} = \frac{a_1}{3!}$$

$$n=2,\ a_4 = \frac{a_2}{4.3} = \frac{a_0}{4.3.2.1} = \frac{a_0}{4!}$$

$$n=3,\ a_5 = \frac{a_3}{5.4} = \frac{a_1}{5.4.3.2.1} = \frac{a_1}{5!}$$

$$n=4,\ a_6 = \frac{a_4}{6.5} = \frac{a_0}{6.5.4.3.2.1} = \frac{a_0}{6!}\ \cdots$$

$$\Rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots$$

$$= a_0 + a_1x + \frac{a_0}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \frac{a_0}{6!}x^6 + \frac{a_1}{7!}x^7 + \cdots$$

$$= a_0 \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)}_{\cosh x} + a_1 \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)}_{\sinh x}$$

$$= a_0 \cosh x + a_1 \sinh x$$

- another method: $y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$\Rightarrow y(x) = a_0 e^{r_1 x} + a_1 e^{r_2 x} = a_0 e^x + a_1 e^{-x} \text{ or}$$

$$= a_0 \cosh x + a_1 \sinh x \quad \text{as expected}$$