Mathematical physics (2)
HW #2 - Solution
Dr. Gassem Alzoubi

① **problem 11.3.2:** simplify the following expression

$$\Gamma(2/3)\,\Gamma(5/3) = \Gamma(2/3)\,\Gamma(\tfrac{2}{3}+1) = \Gamma(2/3)\cdot\tfrac{2}{3}\,\Gamma(2/3) = \tfrac{2}{3}\left[\Gamma(2/3)\right]^2$$

② **problem 11.3.5:** simplify the following expression

$\Gamma(1/2)\,\Gamma(4)\,\Gamma(9/2)$; $\Gamma(4) = 3! = 6$ and

$$\Gamma(9/2) = \Gamma(\tfrac{7}{2}+1) = \tfrac{7}{2}\,\Gamma(7/2) = \tfrac{7}{2}\,\Gamma(\tfrac{5}{2}+1) = \tfrac{7}{2}\cdot\tfrac{5}{2}\,\Gamma(5/2)$$

$$= \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\Gamma(3/2+1) = \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\,\Gamma(3/2) = \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\,\Gamma(1/2+1)$$

$$= \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma(1/2)$$, so

$$\Gamma(1/2)\,\Gamma(4)\,\Gamma(9/2) = \Gamma(1/2)\cdot 6\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma(1/2)$$

$$= \frac{630}{16}\left[\Gamma(1/2)\right]^2$$

Express the following integrals in terms of Γ functions

③ - problem 11.3.10 $\displaystyle\int_0^\infty x^{-2/5} e^{-x}\,dx$

let $p-1 = -2/5 \Rightarrow p = 3/5$

$$\Rightarrow \int_0^\infty x^{-2/5} e^{-x}\,dx = \int_0^\infty x^{\frac{3}{5}-1} e^{-x}\,dx = \Gamma(3/5)$$

- problem 11.3.12 $\int_0^\infty x e^{-x^3} dx$, let $u = x^3$, $du = 3x^2 dx$

$\Rightarrow dx = \frac{du}{3x^2} = \frac{du}{3u^{2/3}}$

$$\Rightarrow \int_0^\infty x e^{-x^3} dx = \int_0^\infty u^{1/3} e^{-u} \frac{du}{3u^{2/3}}$$

$$= \frac{1}{3} \int_0^\infty u^{-1/3} e^{-u} du = \frac{1}{3} \int_0^\infty u^{2/3 - 1} e^{-u} du = \frac{1}{3} \Gamma(2/3)$$

- problem 11.3.14 $\int_0^1 \sqrt[3]{\ln x}\, dx$, let $u = -\ln x$, $x = e^{-u}$

$du = -\frac{1}{x} dx$

$dx = -x\, du$

$= -e^{-u} du$

at $x=0 \Rightarrow u = \infty$

at $x=1$, $u = 0$

$$\Rightarrow \int_0^1 (\ln x)^{1/3} dx = \int_\infty^0 (-u)^{1/3} (-e^{-u}) du$$

$$= \int_0^\infty (-u)^{1/3} e^{-u} du = (-1)^{1/3} \int_0^\infty u^{1/3} e^{-u} du$$

$$= - \int_0^\infty u^{\frac{4}{3}-1} e^{-u} du = -\Gamma(4/3)$$

---

(4) problem 11.5.1: Find ; $\Gamma(p) = \frac{1}{p} \Gamma(p+1)$

- $\Gamma(3/2) = \Gamma(1/2 + 1) = \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

- $\Gamma(-1/2) = \frac{1}{-1/2} \Gamma(-1/2 + 1) = -2 \Gamma(1/2) = -2\sqrt{\pi}$

- $\Gamma(-3/2) = \frac{1}{-3/2} \Gamma(-3/2 + 1) = -\frac{2}{3} \Gamma(-1/2)$

$= -\frac{2}{3} \{-2\sqrt{\pi}\} = \frac{4}{3} \sqrt{\pi}$

⑤ problem 11.5.3: show that $\binom{p}{n} = \frac{\Gamma(p+1)}{n!\,\Gamma(p-n+1)}$

using

$$\binom{p}{n} = \frac{p!}{n!\,(p-n)!} = \frac{\Gamma(p+1)}{n!\,\Gamma(p-n+1)} \checkmark$$

---

⑥ problem 11.5.4: show that $\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$

using $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx$, ~~tot~~

$$\Gamma(n+\tfrac{1}{2}) = \int_0^\infty x^{n+\frac{1}{2}-1} e^{-x}\,dx = \int_0^\infty x^{n-\frac{1}{2}} e^{-x}\,dx$$

let $x = u^2 \Rightarrow dx = 2u\,du \Rightarrow$

$$\Gamma(n+\tfrac{1}{2}) = \int_0^\infty u^{2n-1} e^{-u^2}\, 2u\,du = 2\underbrace{\int_0^\infty u^{2n} e^{-u^2}\,du}$$

standard Gaussian integral

using $\int_0^\infty x^{2n} e^{-\alpha x^2}\,dx = \frac{1}{2}\frac{(2n)!}{4^n n!}\sqrt{\frac{\pi}{\alpha^{2n+1}}}$; with $\alpha = 1$

$$\Rightarrow \int_0^\infty u^{2n} e^{-u^2}\,du = \frac{1}{2}\frac{(2n)!}{4^n n!}\sqrt{\pi}$$

$$\Rightarrow \Gamma(n+\tfrac{1}{2}) = \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

---

⑦ problem 11.5.5:

a) show that $\Gamma(\tfrac{1}{2}-n)\,\Gamma(\tfrac{1}{2}+n) = (-1)^n \pi$; n is a positive integer using $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin \pi p}$ and letting $p = \frac{1}{2} - n$

$$\Gamma(\tfrac{1}{2}-n)\,\Gamma(1-\tfrac{1}{2}+n) = \Gamma(\tfrac{1}{2}-n)\,\Gamma(\tfrac{1}{2}+n) = \frac{\pi}{\sin \pi(\frac{1}{2}-n)}$$

now $\sin(a-b) = \sin a \cos b - \cos a \sin b \Rightarrow$ so

$$\sin\left(\frac{\pi}{2} - n\pi\right) = \sin\frac{\pi}{2}\cos n\pi - \cancel{\cos\frac{\pi}{2}\sin n\pi}^{\,0} = \cos n\pi = (-1)^n$$

$$\therefore \Gamma(\tfrac{1}{2}-n)\,\Gamma(\tfrac{1}{2}+n) = \frac{\pi}{(-1)^n} = (-1)^n \pi$$

b) show that $(z)!\,(-z)! = \dfrac{\pi z}{\sin \pi z}$

now using $\Gamma(z) = (z-1)!$, we have $z! = \Gamma(1+z)$
$-z! = \Gamma(1-z)$

$$\Rightarrow (z)!\,(-z)! = \Gamma(1+z)\,\Gamma(1-z)$$

$$= \frac{\pi z}{\sin \pi z} \quad \text{identity}$$

---

⑧ <u>problem 11.7.2:</u> evaluate $\displaystyle\int_0^{\pi/2} \sqrt{\sin^3\theta \cos\theta}\, d\theta = \int_0^{\pi/2} \sin^{3/2}\theta \cos^{1/2}\theta \, d\theta$

compare with $B(p,q) = 2\displaystyle\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta \Rightarrow$

$2p-1 = 3/2 \Rightarrow 2p = 5/2$
$p = 5/4$

and $2q-1 = 1/2 \Rightarrow 2q = 3/2$
$\Rightarrow q = 3/4$

$$\Rightarrow \int_0^{\pi/2} \sin^{3/2}\theta \cos^{1/2}\theta \, d\theta = \frac{B(5/4, 3/4)}{2}$$

$$= \frac{1}{2}\,\frac{\Gamma(5/4)\,\Gamma(3/4)}{\Gamma(2)} = \frac{1}{2}\cdot\frac{1}{4}\,\Gamma(1/4)\,\Gamma(3/4)$$

$$= \frac{1}{8}\,\Gamma(1/4)\,\Gamma(3/4)$$

$$= \frac{\sqrt{2}}{8}\,\pi$$

but $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin \pi p}$

let $p = 1/4$

$$\Rightarrow \Gamma(1/4)\,\Gamma(3/4) = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\,\pi$$

(9) ~~P~~problem 11.7.3: evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \int_0^1 (1-x^3)^{-1/2} dx, \quad \text{let } u = x^3, \quad du = 3x^2 dx$$

$$dx = \frac{du}{3x^2} = \frac{du}{3u^{2/3}}$$

$$= \int_0^1 (1-u)^{-1/2} \frac{du}{3u^{2/3}} = \frac{1}{3}\int_0^1 u^{-2/3}(1-u)^{-1/2} du$$

compare with $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \Rightarrow$ $p-1 = -2/3 \Rightarrow p = 1/3$

and $q-1 = -1/2 \Rightarrow q = 1/2$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{3} B(1/3, 1/2)$$

---

(10) problem 11.7.6: evaluate $\int_0^\infty \frac{y\,dy}{(1+y^3)^2}$

let $u = y^3 \Rightarrow du = 3y^2 dy$

$$\Rightarrow y\,dy = \frac{du}{3y} = \frac{du}{3u^{1/3}}$$

$$\Rightarrow \int_0^\infty \frac{y\,dy}{(1+y^3)^2} = \int_0^\infty \frac{du}{3u^{1/3}} \frac{1}{(1+u)^2} = \frac{1}{3}\int_0^\infty \frac{u^{-1/3}\,du}{(1+u)^2}$$

compare with $B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}} dy \Rightarrow$ $p-1 = -1/3 \Rightarrow p = 2/3$

$p+q = 2 \Rightarrow q = 2-p = 2-2/3 = 4/3$

$$\therefore \int_0^\infty \frac{y\,dy}{(1+y^3)^2} = \frac{1}{3}\int_0^\infty \frac{u^{-1/3}\,du}{(1+u)^2} = \frac{1}{3} B(2/3, 4/3) = \frac{1}{3} \frac{\Gamma(2/3)\Gamma(4/3)}{\Gamma(2)}$$

$$= \frac{1}{3}\Gamma(2/3)\Gamma(4/3) = \frac{1}{3}\Gamma(2/3)\cdot\frac{1}{3}\Gamma(1/3) = \frac{1}{9}\Gamma(1/3)\Gamma(2/3)$$

using $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$

$$\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$$

$$= \frac{1}{9\sqrt{3}} 2\pi$$

$$= \frac{1}{9\sqrt{3}} 2\pi \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\pi\sqrt{3}}{27}$$

(11) problem 11.7.9: show that $B(n,n) = B(n, 1/2)/2^{2n-1}$

$$B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1} (\cos^2\theta)^{q-1} \underbrace{2\sin\theta\cos\theta}_{\sin 2\theta}\, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{p-1} (\cos^2\theta)^{q-1} \sin 2\theta\, d\theta \text{ ; let } p = q = n$$

$$\Rightarrow B(n,n) = \int_0^{\pi/2} (\sin^2\theta)^{n-1} (\cos^2\theta)^{n-1} \sin 2\theta\, d\theta = \int_0^{\pi/2} [\underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}}]^{2n-2} \sin 2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{[\sin 2\theta]^{2n-2}}{2^{2n-2}} \cdot \sin 2\theta\, d\theta = \frac{1}{2^{2n-2}} \int_0^{\pi/2} \{\sin 2\theta\}^{2n-1} d\theta \quad \times \frac{2}{2}$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1} d(2\theta) \quad \text{relabel } 2\theta \to \phi$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} (\sin\phi)^{2n-1} d\phi \text{ ; compare with } \int_0^{\pi/2} (\sin\phi)^{2p-1} (\cos\phi)^{2q-1} d\phi$$

$$\Rightarrow \quad 2p-1 = 2n-1 \Rightarrow p = n, \quad 2q-1 = 0 \Rightarrow q = 1/2$$

$$= \frac{1}{2^{2n-1}} \beta(n, 1/2) \checkmark$$

NOW $B(n,n) = \frac{1}{2^{2n-1}} B(n, 1/2) = \frac{1}{2^{2n-1}} \frac{\cancel{\Gamma(n)}\, \Gamma(1/2)}{\Gamma(n+1/2)} \Rightarrow$

$$\frac{\Gamma(n)\cancel{\Gamma(n)}}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \frac{\Gamma(1/2)}{\Gamma(n+1/2)}$$

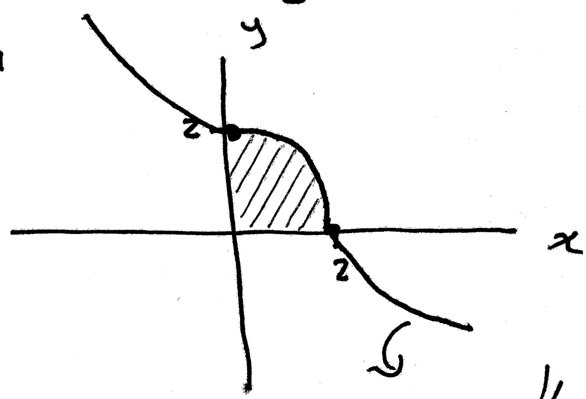
$$\Rightarrow \Gamma(2n) = 2^{2n-1} \frac{\Gamma(n)\Gamma(n+1/2)}{\Gamma(1/2)} = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\Gamma(n+1/2)$$

$$\therefore \Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\,\Gamma(n+1/2)$$

This proves the duplication formula for Γ functions

(12) Problems 11.7.10 and 11.7.11

a) computer plot the graph $x^3+y^3=8$. use any online plotting tools such as WolframAlpha



$y=(8-x^3)^{1/3}$

b) Find the first quadrant area bounded by the curve

$$\text{Area}=A=\int y\,dx=\int_0^2 (8-x^3)^{1/3}dx$$

let $u=x^3$, $du=3x^2dx$, $dx=\frac{du}{3x^2}=\frac{du}{3u^{2/3}}$

at $x=0$, $u=0$

$x=2$, $u=8$

$$\Rightarrow A=\int_0^8 (8-u)^{1/3}\frac{du}{3u^{2/3}}=\frac{1}{3}\underbrace{\int_0^8 u^{-2/3}(8-u)^{1/3}du}$$

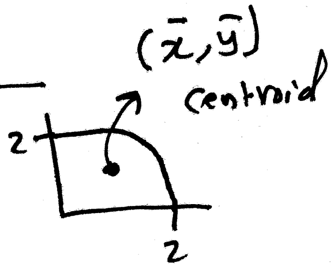
compare with $B(p,q)=\frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}dy$

$\Rightarrow p-1=-2/3 \Rightarrow p=1/3$, $q-1=1/3\Rightarrow q=4/3$, $a=8$

$$\Rightarrow A=\frac{1}{3}(8)^{\frac{1}{3}+\frac{4}{3}-1}\beta(1/3,4/3)=\frac{1}{3}(8)^{2/3}\beta(1/3,4/3)$$

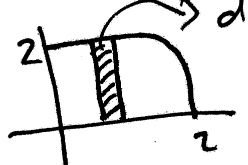
$$=\frac{1}{3}(2^3)^{2/3}\beta(1/3,4/3)=\frac{4}{3}\beta(1/3,4/3)$$

c) Find the centroid of the area



let the centroid be $(\bar{x},\bar{y})$, where

$\bar{x}=\frac{\int_0^2 x\,dA}{A}$ and $\bar{y}=\frac{\int_0^2 y\,dA}{A}$; $A=\frac{4}{3}\beta(1/3,4/3)$



$dA=y\,dx$

$$\Rightarrow \bar{x}=\frac{1}{A}\int_0^2 x(8-x^3)^{1/3}dx$$

, let $u=x^3$, $du=3x^2dx$

$\Rightarrow dx=\frac{du}{3x^2}=\frac{du}{3u^{2/3}}$

$$\Rightarrow \int_0^2 x(8-x^3)^{1/3}\,dx = \int_0^8 u^{1/3}(8-u)^{1/3}\,\frac{du}{3u^{2/3}}$$

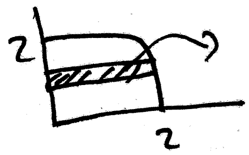
$$= \frac{1}{3}\int_0^\infty u^{-1/3}(8-u)^{1/3}\,du$$

$p-1=-1/3 \Rightarrow p=2/3$

$q-1=1/3 \Rightarrow q=4/3$

$$= \frac{1}{3}\,(8)^{\frac{2}{3}+\frac{4}{3}-1}\,\beta(2/3,4/3) = \frac{1}{3}\,8\,B(2/3,4/3)$$

$$\Rightarrow \bar{x} = \frac{\frac{1}{3}\,8\,B(2/3,4/3)}{\frac{4}{3}\,B(1/3,4/3)} = 2\,\frac{B(2/3,4/3)}{B(1/3,4/3)} = 0.91$$

similarly $\bar{y} = \dfrac{\int_0^2 y\,dA}{A}$;  $dA = x\,dy$

$$\Rightarrow \int_0^2 y\,dA = \int_0^2 y\,x\,dy = \int_0^2 y(8-y^3)^{1/3}\,dy = \frac{8}{3}\,B(2/3,4/3)$$

see that

$$\bar{y} = \frac{\frac{8}{3}\,B(2/3,4/3)}{\frac{4}{3}\,B(1/3,4/3)} = 2\,\frac{B(2/3,4/3)}{B(1/3,4/3)} = 0.91$$

note that due to symmetry $\bar{x}=\bar{y}=0.91$

---

(13) problem 11.11.5: use Stirling's formula to

evaluate $\displaystyle\lim_{n\to\infty} \frac{\Gamma(n+3/2)}{\sqrt{n}\;\Gamma(n+1)}$

using $\Gamma(n+1/2) = \dfrac{(2n)!}{4^n\,n!}\sqrt{\pi}$ and $\Gamma(n+1)=n!$, we get

$$\lim_{n\to\infty} \frac{\Gamma(n+3/2)}{\sqrt{n}\,\Gamma(n+1)} = \lim_{n\to\infty} \frac{\Gamma(n+1/2+1)}{\sqrt{n}\,\Gamma(n+1)}$$

$$= \lim_{n\to\infty} \frac{(n+1/2)\,\Gamma(n+1/2)}{\sqrt{n}\,\Gamma(n+1)} = \lim_{n\to\infty} \frac{(n+1/2)\,\frac{(2n)!}{4^n n!}\sqrt{\pi}}{\sqrt{n}\;n!}$$

$$= \lim_{n\to\infty} \frac{(n+1/2)\,(2n)!\,\sqrt{\pi}}{\sqrt{n}\,4^n\,(n!)^2} \quad ; \text{ Now using } n! \approx n^n e^{-n}\sqrt{2\pi n}$$

$$= \lim_{n\to\infty} \frac{(n+1/2)\,(2n)^{2n}\,\cancel{e^{-2n}}\,\sqrt{4\pi n}\,\sqrt{\pi}}{\sqrt{n}\,4^n\,n^{2n}\,\cancel{e^{-2n}}\,(2\pi n)} = \lim_{n\to\infty} \frac{(n+1/2)\,2^{2n}\,\cancel{n^{2n}}\,\sqrt{4\pi n}\,\sqrt{\pi}}{\sqrt{n}\,(2^2)^n\,\cancel{n^{2n}}\,(2\pi n)}$$

$$= \lim_{n\to\infty} \frac{(n+1/2)\,\cancel{2^{2n}}\,\sqrt{4\pi}\,\cancel{\sqrt{n}}\,\sqrt{\pi}}{\cancel{\sqrt{n}}\,\cancel{2^{2n}}\,(2\pi n)} = \lim_{n\to\infty} \frac{(n+1/2)}{n}$$

$$= \lim_{n\to\infty} \left[1 + \frac{1}{2n}\right] = 1$$

---

(14) problem 8.11.13 using δ functions, write the following mass or charge density functions

a) mass 5 at $x = 2$, and mass 3 at $x = -7$

$\rho = 5\,\delta(x-2) + 3\,\delta(x+7)$, mass density

b) charge 3 at $x = -5$, and charge -4 at $x = 10$

$\rho = 3\,\delta(x+5) - 4\,\delta(x-10)$, charge density

(15) problem 8.11.15: evaluate the following integrals

a) $\int_0^{\pi} \sin x \, \delta(x - \pi/2) \, dx = \sin \frac{\pi}{2} = 1$

b) $\int_0^{\pi} \sin x \, \delta(x + \pi/2) = \int_0^{\pi} \sin x \, \delta(x - (-\pi/2)) dx = 0$

as $x = -\pi/2$ is outside the interval $[0, \pi]$

c) $\int_{-1}^{1} e^{3x} \delta'(x) \, dx = (-1)^1 \frac{d}{dx}\Big|_{x=0} e^{3x} = (-1)^1 \left[3e^{3x} \right]\Big|_{x=0}$

$= -3$

d) $\int_0^{\pi} \cosh x \, \delta''(x-1) \, dx = (-1)^2 \frac{d^2}{dx^2}\Big|_{x=1} \cosh x = \cosh(1)$

(16) problem 8.11.21(d): evaluate $\int_{-\pi/2}^{\pi/2} \cos x \, \delta(\sin x) \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$\Rightarrow \int_{-1}^{1} \cancel{\cos x} \, \delta(u) \frac{du}{\cancel{\cos x}} = \int_{-1}^{1} \delta(u) du = 1$

problem

(17) show that $x^2 \delta'(x) = 0$

it can be proved in two ways

i) 1^{st} method : let us find the following integral

$\int_{-\infty}^{\infty} x^2 \delta'(x)\, dx$; integrating by parts

$u = x^2$, $dv = \delta'(x)$

$du = 2x\,dx$, $v = \delta(x)$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 \delta'(x) = \underbrace{x^2 \delta(x) \Big|_{-\infty}^{\infty}}_{\text{zero as } \delta(\infty) = \delta(-\infty) = 0} - \underbrace{2\int_{-\infty}^{\infty} x\,\delta(x)\, dx}_{= x\big|_{x=0} = 0}$$

$= 0 \quad \Rightarrow \boxed{x^2 \delta'(x) = 0}$

ii) 2^{nd} method : using the identity

$\phi(x)\, \delta^{(n)}(x) = (-1)^n \delta(x)\, \phi^{(n)}(x)$, let $\phi(x) = x^2$, and

consider first derivative $(n=1) \Rightarrow$

$$x^2 \delta'(x) = (-1)^1 \delta(x) \frac{d}{dx} x^2 = -\delta(x)\, 2x$$

$= -2x\,\delta(x) = 0$ as $x\,\delta(x) = 0$ always

$\therefore \boxed{x^2 \delta'(x) = 0}$

---

(18) problem 8.11.18 (d):

show that $(x^2+y^2+z^2)\, \nabla^2 (\delta(x)\delta(y)\delta(z)) = 6\,\delta(x)\delta(y)\delta(z)$

$$\Rightarrow (x^2+y^2+z^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Big(\delta(x)\delta(y)\delta(z)\Big) =$$

$$\underbrace{(x^2+y^2+z^2)\frac{\partial^2}{\partial x^2}\delta(x)\delta(y)\delta(z)}_{\mathrm{I}} + \underbrace{(x^2+y^2+z^2)\frac{\partial^2}{\partial y^2}(\delta(x)\delta(y)\delta(z))}_{\mathrm{II}} + \underbrace{(x^2+y^2+z^2)\frac{\partial^2}{\partial z^2}(\delta(x)\delta(y)\delta(z))}_{\mathrm{III}}$$

$$I = (x^2+y^2+z^2)\frac{\partial^2}{\partial x^2}\,\delta(x)\delta(y)\delta(z) = (x^2+y^2+z^2)\,\delta(y)\delta(z)\delta''(x)$$

$$= x^2\delta(y)\delta(z)\delta''(x) + \underbrace{y^2\delta(y)\delta(z)\delta''(x)}_{0 \text{ as } y^2\delta(y)=0} + \underbrace{z^2\delta(y)\delta(z)\delta''(x)}_{0 \text{ as } z^2\delta(z)=0}$$

$$= \delta(y)\delta(z)\,x^2\delta''(x)$$

similarly $II = \delta(x)\delta(z)\,y^2\delta''(y)$; $III = \delta(x)\delta(y)\,z^2\delta''(z)$

$\Rightarrow$ Now using $x^n\delta^{(n)}(x) = (-1)^n\, n!\,\delta(x)$

$$x^n\delta''(x) = (-1)^2\,2!\,\delta(x) = 2\delta(x)$$

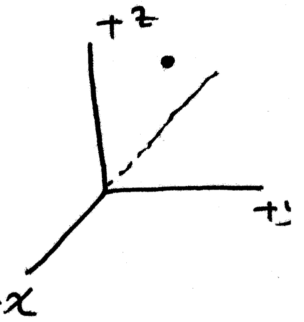
$$\Rightarrow I = \delta(y)\delta(z)\,x^2\delta''(x) = 2\,\delta(x)\delta(y)\delta(z)$$

similarly $II = 2\delta(x)\delta(y)\delta(z)$ and $III = 2\delta(x)\delta(y)\delta(z)$

$$\Rightarrow I + II + III = 6\,\delta(x)\delta(y)\delta(z) \checkmark$$

---

problem 8.11.23(c) write the density of point charge $(q=1)$ or point mass $(m=1)$ located at $(-2, 0, 2\sqrt{3})$



in cartesian $\Rightarrow \rho(x,y,z) = \delta(x+2)\delta(y)\delta(z-2\sqrt{3})$

in cylindrical $r = \sqrt{x^2+y^2} = \sqrt{4+0} = 2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \theta = 0$; but this is with $-x$-axis $\Rightarrow \theta = \pi$ with positive $+x$-axis

$$\Rightarrow \rho(r,\theta,z) = \frac{\delta(r-2)\,\delta(\theta-\pi)\,\delta(z-2\sqrt{3})}{r}$$

in spherical $\Rightarrow (r,\theta,\phi)$

$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+12} = \sqrt{16} = 4$, $z = r\cos\theta \Rightarrow \cos\theta = \frac{z}{r} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = 30^\circ = \pi/6$, $\phi = \pi$

$$\Rightarrow \rho(r,\theta,\phi) = \frac{\delta(r-4)\,\delta(\theta-\pi/6)\,\delta(\phi-\pi)}{r^2\sin\theta}$$