Mathematical Physics (2)
HW #10 - solution
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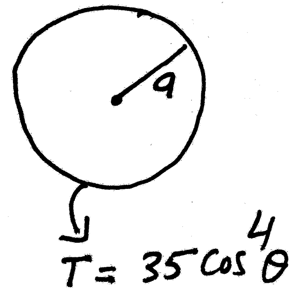
① <u>problem 13.7.1:</u> Find the steady-state temperature distribution inside and outside a sphere of radius 1 $(a=1)$ when the surface temperature is given by

$T(\theta) = 35\cos^4\theta$



see that the surface temperature is independent on ϕ (azimuthal symmetry)
$\Rightarrow m=0$, so the general solution is

$$T(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

a) inside the sphere $(r<1)$; $B_l = 0$

$$\Rightarrow T(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$; now at $r=1$, we have

$$T(1,\theta) = \sum_{l=0}^{\infty} A_l P_l(\cos\theta) \quad ---- (1)$$

$$35\cos^4\theta = \sum_{l=0}^{\infty} A_l P_l = A_0P_0 + A_1P_1 + A_2P_2 + A_3P_3 + A_4P_4 + \cdots \quad --- (2)$$

now express $\cos^4\theta$ in terms of Legendre Polynomials

$P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, $P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$

from $P_2 \Rightarrow 2P_2 = 3x^2 - 1 \Rightarrow x^2 = \frac{2}{3}P_2 + \frac{1}{3}$

$\Rightarrow$

and from P_4, we have
$$\begin{aligned} 8P_4 &= 35x^4 - 30x^2 + 3 \\ &= 35x^4 - 20P_2 - 10 + 3 \\ &= 35x^4 - 20P_2 - 7 \quad ; \text{ but } P_0 = 1 \\ &= 35x^4 - 20P_2 - 7P_0 \end{aligned}$$

$\Rightarrow 35x^4 = 8P_4 + 20P_2 + 7P_0$, substitute in (2), we get

$8P_4 + 20P_2 + 7P_0 = A_0P_0 + A_1P_1 + A_2P_2 + A_3P_3 + A_4P_4 + A_5P_5 + \cdots$

equate coefficients $\Rightarrow$

$A_0 = 7,\ A_1 = 0,\ A_2 = 20,\ A_3 = 0,\ A_4 = 8,\ A_5 = A_6 = \cdots = 0$

$$\Rightarrow T(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = A_0P_0 + A_2r^2P_2 + A_4r^4P_4$$

$$= 7P_0 + 20r^2P_2 + 8r^4P_4$$

$$= 7P_0(\cos\theta) + 20r^2P_2(\cos\theta) + 8r^4P_4(\cos\theta)$$

b) outside the spher $(r>1)$

$T(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$; at the surface $r=1$

$$T(1,\theta) = \sum_{l=0}^{\infty} B_l P_l(\cos\theta) = B_0P_0 + B_1P_1 + B_2P_2 + B_3P_3 + \cdots$$

$$+ B_4P_4 + B_5P_5 + \cdots$$

$$35\cos^4\theta = B_0P_0 + B_1P_1 + B_2P_2 + B_3P_3 + B_4P_4 + B_5P_5 + \cdots$$

$8P_4 + 20P_2 + 7P_0 = B_0P_0 + B_1P_1 + B_2P_2 + B_3P_3 + B_4P_4 + \cdots$

equate coefficients $B_0 = 7,\ B_1 = 0,\ B_2 = 20,\ B_3 = 0,\ B_4 = 8,$

$B_5 = B_6 = B_7 = \cdots = 0$

$$\Rightarrow T(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) = \frac{B_0}{r}P_0 + \frac{B_2}{r^3}P_2 + \frac{B_4}{r^5}P_4$$

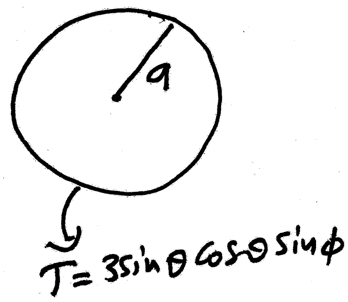
$$= \frac{7}{r}P_0 + \frac{20}{r^3}P_2 + \frac{8}{r^5}P_4$$

Note that both solutions inside and outside match at the surface $(r=1)$ as Temperature is contineous around the surface of the sphere.

i.e $T(1,\theta)\big|_{\text{inside}} = T(1,\theta)\big|_{\text{outside}}$

② problem 13.7.9: Find the steady-state temperature distribution inside and outside a sphere of radius 1 ($a=1$) when the surface temperature is given by

$T(\theta,\phi) = 3\sin\theta\cos\theta\sin\phi$



Here T depends on both θ and ϕ

(No azimuthal symmetry $\Rightarrow m \neq 0$) $\Rightarrow$

the general solution is given by

$$T(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(A_{lm} r^{l} + \frac{B_{lm}}{r^{l+1}}\right) Y_{lm}(\theta,\phi)$$

a) inside the sphere $(r<1)$; $B_{lm} = 0 \Rightarrow$

$$T(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm} r^{l} Y_{lm}(\theta,\phi) \quad \text{--(1)}$$

, now at the surface $r=1$

$$T(1,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm} Y_{lm}$$

$$\underbrace{3\sin\theta\cos\theta\sin\phi}_{} = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm} Y_{lm}; \quad \text{--(2)}$$

let us express this in terms of spherical harmonics

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta(\cos\phi + i\sin\phi) \quad \text{--(3)}$$

$$Y_{2,-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta(\cos\phi - i\sin\phi) \quad (4)$$

add (3) + (4) $\Rightarrow$ $\sin\theta\cos\theta\sin\phi = -\frac{1}{2i}\sqrt{\frac{8\pi}{15}}\left[Y_{21} + Y_{2,-1}\right]$

substitute in (1), we get

$$-\frac{3}{2i}\sqrt{\frac{8\pi}{15}}\left[Y_{2,1} + Y_{2,-1}\right] = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} Y_{lm}$$

Clearly, the only terms that servive having $l=2$ and $m=\pm 1$; i.e $A_{2,1}$ and $A_{2,-1}$

$$\Rightarrow \frac{-3}{2i}\sqrt{\frac{8\pi}{15}}\,Y_{2,1} - \frac{3}{2i}\sqrt{\frac{8\pi}{15}}\,Y_{2,-1} = A_{21}Y_{2,1} + A_{2,-1}Y_{2,-1}$$

equate coefficients $\Rightarrow A_{21} = A_{2,-1} = \frac{-3}{2i}\sqrt{\frac{8\pi}{15}}$

$\Rightarrow$ from (1), we have

$$T(r,\theta,\phi) = A_{21}r^2 Y_{21} + A_{2,-1}r^2 Y_{2,-1}$$

$$= \frac{-3}{2i}\sqrt{\frac{8\pi}{15}}\,r^2\{Y_{2,1}+Y_{2,-1}\} = \frac{-3}{2i}\sqrt{\frac{8\pi}{15}}\,r^2\left[-2i\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\sin\phi\right]$$

$$= 3r^2\sin\theta\cos\theta\sin\phi \quad ; \text{ using } P_2^1 = 3\sin\theta\cos\theta$$

$$= r^2 P_2^1(\cos\theta)\sin\phi$$

b) outside the sphere $(r>1)$;

$$T(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{B_{lm}}{r^{l+1}}Y_{lm} \quad ; \text{ at surface } (r=1) \quad \text{--- (5)}$$

$$T(1,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}Y_{lm}, \quad \text{--- (6)}$$

$$3\sin\theta\cos\theta\sin\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}Y_{lm}$$

$$\frac{-3}{2i}\sqrt{\frac{8\pi}{15}}\,Y_{2,1} - \frac{3}{2i}\sqrt{\frac{8\pi}{15}}\,Y_{2,-1} = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}B_{lm}Y_{lm}$$

again two terms survive with $l=2$ and $m=\pm1$ $(B_{21}, B_{2,-1})$

$\Rightarrow$ equate coefficients $\Rightarrow B_{2,1} = B_{2,-1} = \frac{-3}{2i}\sqrt{\frac{8\pi}{15}}$

from (5) $T(r,\theta,\phi) = \frac{B_{21}}{r^3}Y_{2,1} + \frac{B_{2,-1}}{r^3}Y_{2,-1}$

$$= \frac{-3}{2i}\sqrt{\frac{8\pi}{15}}\,\frac{1}{r^3}\left[Y_{2,1}+Y_{2,-1}\right] = \frac{3}{r^3}\sin\theta\cos\theta\sin\phi$$

$$= \frac{1}{r^3}(3\sin\theta\cos\theta)\sin\phi$$

$$= \frac{1}{r^3}P_2^1(\cos\theta)\sin\phi$$

Note that at the surface $(r=1)$, both solutions inside and outside match as expected.