Mathematical physics (2)
HW # 1 - solution
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① problem 10.8.1: in spherical coordinates, Find the scale factors h_r, h_θ, h_ϕ, $d\vec{s}$, unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ and write the g_{ij} matrix, dV the volume element

- $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ $\quad (q_1, q_2, q_3) \downarrow (r, \theta, \phi)$

$$h_{11}^2 = h_1^2 = h_r^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta = \sin^2\theta[\cos^2\phi + \sin^2\phi] + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1 \Rightarrow \boxed{h_r = 1}$$

$$h_{22}^2 = h_2^2 = h_\theta^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta$$

$$= r^2\cos^2\theta\,\underbrace{[\cos^2\phi + \sin^2\phi]}_{1} + r^2\sin^2\theta$$

$$= r^2\cos^2\theta + r^2\sin^2\theta = r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = r^2$$

$$\Rightarrow \boxed{h_\theta = r}$$

$$h_{33}^2 = h_3^2 = h_\phi^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi$$

$$= r^2\sin^2\theta\,\underbrace{\{\sin^2\phi + \cos^2\phi\}}_{1} = r^2\sin^2\theta$$

$$\Rightarrow \boxed{h_\phi = r\sin\theta}$$

$$\Rightarrow g_{ij} = \begin{pmatrix} h_{11}^2 & h_{12}^2 & h_{13}^2 \\ h_{21}^2 & h_{22}^2 & h_{23}^2 \\ h_{31}^2 & h_{32}^2 & h_{33}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Note that the off-diagonal elements are zeros as the spherical coordinate system is an orthogonal system. one can prove that all $h_{ij} = 0$ for $i \neq j$, for example $h_{12}^2 = h_{r\theta}^2 = \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right)$

$$= r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\sin\theta\cos\theta$$

$$= r\sin\theta\cos\theta\,\underbrace{(\cos^2\phi + \sin^2\phi)}_{1} - r\sin\theta\cos\theta$$

$$= r\sin\theta\cos\theta - r\sin\theta\cos\theta = \text{zero}$$

similarly for all other off-diagonal elements

- $d\vec{s} = h_r\,dr\,\hat{r} + h_\theta\,d\theta\,\hat{\theta} + h_\phi\,d\phi\,\hat{\phi}$
  $= dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$
- $dV = h_r h_\theta h_\phi\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$
- to find unit vectors, let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow$

$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k} \Rightarrow$

$\hat{r} = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{\theta} = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\phi\,\hat{k}$

$\hat{\phi} = \frac{1}{h_\phi}\frac{\partial \vec{r}}{\partial \phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

Note that these unit vectors are orthogonal for example $\hat{r}\cdot\hat{\theta} = 0$, $\hat{r}\cdot\hat{\phi} = 0$, $\hat{\theta}\cdot\hat{\phi} = 0$, ----

② <u>problem 10.8.8:</u> consider the parabolic coordinates system (u, v, ϕ); with $x = uv\cos\phi$, $y = uv\sin\phi$, and $z = \frac{1}{2}(u^2 - v^2)$

Find the scale factors

$$h_u^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = v^2\cos^2\phi + v^2\sin^2\phi + u^2 = v^2 + u^2$$

$$\Rightarrow h_u = \sqrt{v^2 + u^2}$$

$$h_v^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = u^2\cos^2\phi + u^2\sin^2\phi + v^2 = u^2 + v^2$$

$$\Rightarrow h_v = \sqrt{u^2 + v^2} = h_u$$

$$h_\phi^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = u^2v^2\sin^2\phi + u^2v^2\cos^2\phi = u^2v^2$$

$$\Rightarrow h_\phi = uv$$

- $d\vec{s} = h_u\, du\, \hat{u} + h_v\, dv\, \hat{v} + h_\phi\, d\phi\, \hat{\phi}$
$= \sqrt{v^2+u^2}\, du\, \hat{u} + \sqrt{v^2+u^2}\, dv\, \hat{v} + uv\, d\phi\, \hat{\phi}$

- $dV = d\tau = h_u h_v h_\phi\, du\, dv\, d\phi = (u^2+v^2)\, uv\, du\, dv\, d\phi$

- unit vectors $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$= uv\cos\phi\,\hat{i} + uv\sin\phi\,\hat{j} + \frac{1}{2}(u^2 - v^2)\,\hat{k}$

$$\Rightarrow \hat{u} = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u} = \frac{1}{\sqrt{u^2+v^2}}\left[v\cos\phi\,\hat{i} + v\sin\phi\,\hat{j} + u\,\hat{k}\right]$$

$$\hat{v} = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v} = \frac{1}{\sqrt{u^2+v^2}}\left[u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} - v\,\hat{k}\right]$$

$$\hat{\phi} = \frac{1}{h_\phi}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{uv}\left[-uv\sin\phi\,\hat{i} + uv\cos\phi\,\hat{j}\right]$$

$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$\Rightarrow$ see that the system is orthogonal as $\hat{u}\cdot\hat{v} = \hat{u}\cdot\hat{\phi} = \hat{v}\cdot\hat{\phi} = \text{zero}$

$$\Rightarrow g_{ij} = \begin{pmatrix} v^2+u^2 & 0 & 0 \\ 0 & v^2+u^2 & 0 \\ 0 & 0 & u^2v^2 \end{pmatrix}$$

③ problem 10.8.15: consider (u,v) coordinates system in two dimensions with $x = u+v$ and $y = v$;

$$h_{11}^2 = h_{uu}^2 = h_u^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 = 1 + 0 = 1 \Rightarrow h_{11} = 1$$

$$h_{22}^2 = h_{vv}^2 = h_v^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = 1 + 1 = 2 \Rightarrow h_{22} = \sqrt{2}$$

$$g_{ij} = \begin{pmatrix} h_{11}^2 & h_{12}^2 \\ h_{21}^2 & h_{22}^2 \end{pmatrix} = \begin{pmatrix} h_{uu}^2 & h_{uv}^2 \\ h_{vu}^2 & h_{vv}^2 \end{pmatrix}$$

$$h_{12}^2 = h_{uv}^2 = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) = (1)(1) + (0)(1) = 1 \Rightarrow h_{12} = 1$$

$$h_{21}^2 = h_{vu}^2 = \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial y}{\partial v}\right)\left(\frac{\partial y}{\partial y}\right) = (1)(1) + (1)(0) = 1 \Rightarrow h_{21} = 1$$

$$\Rightarrow g_{ij} = \begin{pmatrix} h_{11}^2 & h_{12}^2 \\ h_{21}^2 & h_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

, note that $g_{ij} \neq 0$ for $i \neq j \Rightarrow$ the system is not orthogonal and g_{ij} is symmetric as $h_{12}^2 = h_{21}^2$

- unit vectors: $\vec{r} = x\hat{i} + y\hat{j} = (u+v)\hat{i} + v\hat{j}$

$$\hat{u} = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u} = \frac{1}{1}[1\hat{i}] = \hat{i}$$

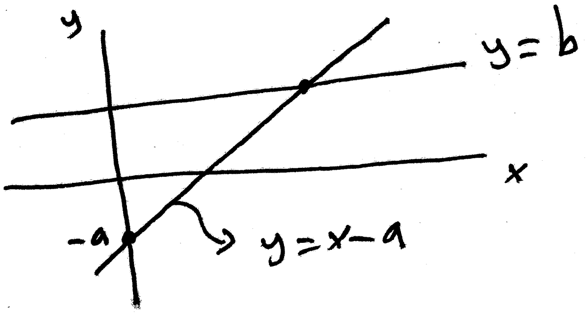
$$\hat{v} = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v} = \frac{1}{\sqrt{2}}[\hat{i} + \hat{j}]$$

$\Rightarrow$ see that $\hat{u}\cdot\hat{v} \neq 0$ as the system is not orthogonal

- if $u = a =$ constant and $v = b =$ constant $\Rightarrow$

$x = a + y \Rightarrow y = x - a$,] two lines

$y = b \Rightarrow y = b$



See that the two lines are not normal at the intersection

④ problem 10.9.16: in cylindrical coordinates system, Find $\nabla \times \hat{\theta}$

$$\nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{q}_1 & h_2 \hat{q}_2 & h_3 \hat{q}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} ; \quad \vec{V} = V_1 \hat{q}_1 + V_2 \hat{q}_2 + V_3 \hat{q}_3$$

$\Rightarrow$ let $\vec{V} = \hat{\theta} = o(\hat{r}) + \hat{\theta} + o(\hat{k})$; $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$

$$\nabla \times \hat{\theta} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ 0 & r & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{r}(0-0) - r\hat{\theta}(0-0) + \hat{k}(1-0)\right] = \frac{1}{r}\hat{k}$$

- Find $\nabla \cdot \hat{r}$; let $\vec{V} = \hat{r} + (0)\hat{\theta} + (0)\hat{k}$; $V_1 = 1$, $V_2 = V_3 = 0$

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_1 h_3 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right]$$

$$\Rightarrow \nabla \cdot \hat{r} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r)\right] = \frac{1}{r}$$

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⑤ problem 10.9.17: in spherical coordinates system

Find

$$\nabla \cdot \hat{r} = \frac{1}{r^2 \cancel{\sin\theta}} \frac{\partial}{\partial r}(r^2 \cancel{\sin\theta}) = \frac{2}{r} ; \quad V_1 = 1,\ V_2 = V_3 = 0$$

$$\nabla \cdot \hat{\theta} = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}(r\sin\theta) = \frac{1}{r\sin\theta}\cos\theta = \frac{1}{r}\cot\theta ; \quad V_2 = 1,\ V_1 = V_3 = 0$$

$$\nabla \times \hat{\theta} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ 0 & r & 0 \end{vmatrix} = \frac{1}{r}\hat{\phi}$$

$$\nabla\times\hat{\phi} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta \end{vmatrix} \quad ; \quad \begin{matrix} V_1 = V_2 = 0 \\ V_3 = 1 \end{matrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\,(r\cos\theta - 0) - r\hat{\theta}\,(\sin\theta - 0) + r\sin\theta\,\hat{\phi}\,(0-0)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[r\cos\theta\,\hat{r} - r\sin\theta\,\hat{\theta}\right] = \frac{1}{r}\left[\cot\theta\,\hat{r} - \hat{\theta}\right]$$

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⑥ problem 10.9.18: in cylindrical system, Find

$$\underline{\nabla\times(\hat{k}\ln r)} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & 0 & \ln r \end{vmatrix} \quad ; \quad \begin{matrix} \vec{V} = V_1\hat{r} + V_2\hat{\theta} + V_3\hat{k} \\ V_1 = V_2 = 0 \\ V_3 = \ln r \end{matrix}$$

$$= \frac{1}{r}\left[\hat{r}\,(0-0) - r\hat{\theta}\left(\frac{1}{r} - 0\right) + \hat{k}\,(0)\right] = -\frac{1}{r}\hat{\theta}$$

- $\underline{\nabla\cdot(r\hat{r} + z\hat{k})}$; $V_1 = r,\ V_2 = 0,\ V_3 = z$

$$\nabla\cdot(r\hat{r} + z\hat{k}) = \frac{1}{r}\left[\frac{\partial}{\partial r}(rr) + \frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial z}(rz)\right]$$

$$= \frac{1}{r}\left[2r + r\right] = \frac{1}{r}\left[3r\right] = 3$$

- $\nabla\ln r = ??$, using

$$\nabla u = \frac{1}{h_1}\frac{\partial u}{\partial q_1}\hat{q}_1 + \frac{1}{h_2}\frac{\partial u}{\partial q_2}\hat{q}_2 + \frac{1}{h_3}\frac{\partial u}{\partial q_3}\hat{q}_3 \quad ; \quad u = \ln r$$

$$\begin{cases} q_1 = r \\ q_2 = \theta \\ q_3 = z \end{cases}$$

$$\Rightarrow \nabla\ln r = \frac{1}{1}\frac{\partial}{\partial r}\ln r\,\hat{r} = \frac{1}{r}\hat{r} \quad ; \quad \hat{r} = \frac{\vec{r}}{r}$$

⑦ problem 10.9.19: in spherical system, find

$\nabla \times (r\hat{\theta})$; $\vec{V} = V_1 \hat{r} + V_2 \hat{\theta} + V_3 \hat{\phi}$; $V_1 = V_3 = 0$, $V_2 = r$

$$\Rightarrow \nabla \times (r\hat{\theta}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & r^2 & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}(0) - r\hat{\theta}(0) + r\sin\theta\,\hat{\phi}(2r-0)\right] = 2\hat{\phi}$$

$$- \nabla(r\cos\theta) = \frac{1}{1}\frac{\partial}{\partial r}(r\cos\theta)\,\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}(r\cos\theta)\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\cos\theta\ r\sin\theta)\Big]$$

$$= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

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⑧ problem 10.9.20: in cylindrical system, find $\nabla^2 r$

using $$\nabla^2 u = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial u}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial u}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial u}{\partial q_3}\right)\right]$$

$$\underline{\underline{\nabla^2 r}} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial r}{\partial r}\right)\right] = \frac{1}{r}$$

$$\underline{\underline{\nabla^2 \ln r}} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln r\right)\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(1)\right] = 0$$

$$\nabla^2\frac{1}{r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) \neq \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\left(-\frac{1}{r}\right)\right)\right)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r}\left(-\frac{1}{r^2}\right) = \frac{1}{r^3} = r^{-3}$$

⑨ <u>problem 10.9.21:</u> in spherical system, find

- $$\nabla^2 r^2 = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial}{\partial r}(r^2)\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}(2r^3)$$

$$= \underline{\frac{1}{r^2}(6r^2) = 6}$$

- $$\nabla^2 e^{ikr\cos\theta} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial}{\partial r} e^{ikr\cos\theta}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta} e^{ikr\cos\theta}\right) + 0\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\sin\theta\, r^2 . ik\cos\theta e^{ikr\cos\theta}\right) + \frac{\partial}{\partial\theta}\left(-\sin^2\theta . ikr e^{ikr\cos\theta}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[ik\sin\theta\cos\theta \frac{\partial}{\partial r}\left(r^2 e^{ikr\cos\theta}\right) - ikr\frac{\partial}{\partial\theta}\left(\sin^2\theta\, e^{ikr\cos\theta}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\Big[ik\sin\theta\cos\theta\left[2re^{ikr\cos\theta} + ik\cos\theta\, r^2 e^{ikr\cos\theta}\right] - ikr\left[2\sin\theta\cos\theta\, e^{ikr\cos\theta} - \sin^3\theta\, ikr e^{ikr\cos\theta}\right]\Big]$$

$$= \frac{e^{ikr\cos\theta}}{r^2\sin\theta}\left\{\cancel{2rik\sin\theta\cos\theta} - r^2k^2\sin\theta\cos^2\theta - \cancel{2ikr\sin\theta\cos\theta} - k^2r^2\sin^3\theta\right\}$$

$$= e^{ikr\cos\theta}\left[-k^2\cos^2\theta - k^2\sin^2\theta\right]$$

$$= -k^2 e^{ikr\cos\theta}\left[\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right] = -k^2 e^{ikr\cos\theta}$$

(10) express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical coordinates

solution: equate $\vec{\nabla}_{xyz} = \vec{\nabla}_{r\theta\phi}$

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

now express $\hat{r}, \hat{\theta}, \hat{\phi}$ on R.H.S in terms of $\hat{i}, \hat{j}, \hat{k}$

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \sin\theta\cos\phi\frac{\partial}{\partial r}\hat{i} + \sin\theta\sin\phi\frac{\partial}{\partial r}\hat{j} + \cos\theta\frac{\partial}{\partial r}\hat{k}$$

$$+\frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}\hat{i} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta}\hat{j} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\hat{k}$$

$$-\frac{1}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\hat{i} + \frac{1}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\hat{j}$$

equate coefficients of unit vectors $\Longrightarrow$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

Note that $-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\phi} = L_z$

complex

when L_z: the z-component of orbital angular momentum $L_z = -i\hbar\frac{\partial}{\partial\phi}$

(11) the angular momentum operator is $\vec{L} = -i\hbar \vec{r} \times \vec{\nabla}$.
find L_x, L_y, L_z in Cartesian coordinates

$$\vec{L} = -i\hbar \vec{r} \times \vec{\nabla} = -i\hbar \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix}$$

$$= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\hat{i} - i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) - i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$= L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$, where

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right);\ L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right);\ L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

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(12) express $\vec{L} = -i\hbar \vec{r} \times \vec{\nabla}$ in spherical coordinates

$$\vec{L} = \frac{-i\hbar}{r^2\sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \end{vmatrix}$$

; $\vec{r} = r\hat{r}$ and $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$

where I used $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

$$\Rightarrow \vec{L} = \frac{-i\hbar}{r^2\sin\theta}\left[\hat{r}(0) - r\hat{\theta}\left(r\frac{\partial}{\partial \phi} - 0\right) + \hat{\phi}\, r^2\sin\theta\frac{\partial}{\partial \theta}\right]$$

$$= \frac{-i\hbar}{r^2\sin\theta}\left[-\hat{\theta}\, r^2\frac{\partial}{\partial \phi} + \hat{\phi}\, r^2\sin\theta\frac{\partial}{\partial \theta}\right]$$

$$= i\hbar\left[\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi} - \hat{\phi}\frac{\partial}{\partial \theta}\right]$$

Now expressing $\hat{\theta}$ and $\hat{\phi}$ in R.H.S in terms of $\hat{i}, \hat{j}, \hat{k}$

$$\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = i\hbar\left[\hat{i}\cot\theta\cos\phi\frac{\partial}{\partial\phi} + \hat{j}\cot\theta\sin\phi\frac{\partial}{\partial\phi} - \hat{k}\frac{\partial}{\partial\phi} + \hat{i}\sin\phi\frac{\partial}{\partial\theta} - \hat{j}\cos\phi\frac{\partial}{\partial\theta}\right]$$

equating coefficients of unit vectors, we get

$$L_x = i\hbar\left[\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$$

$$L_y = i\hbar\left[-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$L_z = i\hbar\left[-\frac{\partial}{\partial\phi}\right] = -i\hbar\frac{\partial}{\partial\phi}$$

define $L_\pm = L_x \pm iL_y$, raising and lowering operators

$$L_+ = L_x + iL_y = \hbar\left[i\sin\phi\frac{\partial}{\partial\theta} + i\cot\theta\cos\phi\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[(\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} + i\left(\cot\theta\cos\phi\frac{\partial}{\partial\phi} + i\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\right]$$

$$= \hbar\left[e^{i\phi}\frac{\partial}{\partial\theta} + ie^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\right] = \hbar e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]$$

similarly $L_- = L_x - iL_y = -\hbar e^{-i\phi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right]$

Now $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = -r^2\nabla^2 + \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

arriving to the last line takes long calculations