Mathematical Physics (1)
HW #9 - solution
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① problem 8.2.6: Solve $y' = \frac{2xy^2+x}{x^2y-y}$ ; $y=0$ when $x=\sqrt{2}$

$\frac{dy}{dx} = \frac{x(2y^2+1)}{y(x^2-1)} \Rightarrow \frac{y}{2y^2+1}dy = \frac{x}{x^2-1}dx$

$\Rightarrow \frac{1}{4}\frac{4y}{2y^2+1}dy = \frac{1}{2}\frac{2x}{x^2-1}dx \Rightarrow$ integrate

$\frac{1}{4}\ln(2y^2+1) = \frac{1}{2}\ln(x^2-1) + A$ ; A constant of integration

multiply by 4 $\Rightarrow \ln(2y^2+1) = 2\ln(x^2-1) + 4A$, Now when $x=\sqrt{2}$ $\Rightarrow y=0$

$\Rightarrow \ln(1) = 2\ln(1) + 4A \Rightarrow 0 = 4A \Rightarrow A = 0$

$\Rightarrow \ln(2y^2+1) = 2\ln(x^2-1) = \ln(x^2-1)^2 \Rightarrow 2y^2+1 = (x-1)^2$

or $\Rightarrow 2y^2 = (x-1)^2 - 1 \Rightarrow \boxed{y^2 = \frac{1}{2}\left[(x-1)^2-1\right]}$

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② problem 8.2.17: speed of particle on the x axis, $x>0$ is $v = \sqrt{x}$. Find $x(t)$ using $t=0, x=0$

$v = \frac{dx}{dt} = \sqrt{x} \Rightarrow \frac{dx}{\sqrt{x}} = dt \Rightarrow x^{-1/2}dx = dt \Rightarrow$ integrate

$2x^{1/2} = t + A \Rightarrow$ at $t=0, x=0 \Rightarrow 0 = 0 + A \Rightarrow A = 0$

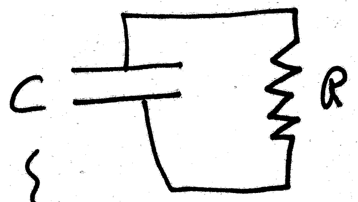
$\Rightarrow 2x^{1/2} = t \Rightarrow 4x = t^2 \Rightarrow x = \frac{1}{4}t^2$ ✓

Now if the particle remains stationary at the origin for a period of time $t_0$ and then starts to move to the right, then for $t > t_0$, we have $x(t) = \frac{1}{4}(t-t_0)^2$ ✓

③ problem 8.2.20:

a) discharging capacitor into Resistor

Find $q(t)$




$\mathcal{E} = V_c + V_R$ ; but $\mathcal{E} = 0 \Rightarrow 0 = IR + \frac{q}{C}$ $\Bigg\}$ at $t=0$, $q = q_0$

$\Rightarrow R\frac{dq}{dt} = -\frac{q}{C} \Rightarrow \frac{dq}{dt} = -\frac{q}{RC}$

$\Rightarrow \frac{dq}{q} = -\frac{dt}{RC}$, integrate $\int_{q_0}^{q} \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt \Rightarrow$

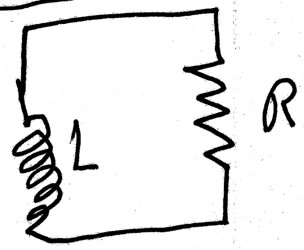
$\Rightarrow \ln\frac{q}{q_0} = \frac{-t}{RC} + A$, Now at $t=0$, $q = q_0 \Rightarrow A = 0$

$\Rightarrow \ln\frac{q}{q_0} = -\frac{t}{RC} \Rightarrow \frac{q}{q_0} = e^{-t/RC} \Rightarrow q(t) = q_0 e^{-t/RC}$

$= q_0 e^{-t/\tau}$

with time constant $\rightarrow \tau = RC$

b) discharging inductor through resistor




again $\mathcal{E} = V_R + V_L$ ; with $\mathcal{E} = 0$

$0 = IR + L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} = -\frac{R}{L}I \Rightarrow \frac{dI}{I} = -\frac{R}{L}dt$

$\Rightarrow \int_{I_0}^{I}\frac{dI}{I} = -\frac{R}{L}\int_0^t dt \Rightarrow \ln\frac{I}{I_0} = -\frac{R}{L}t + A$

Now at $t=0$, $I = I_0 \Rightarrow A = 0 \Rightarrow \ln\frac{I}{I_0} = -\frac{R}{L}t$

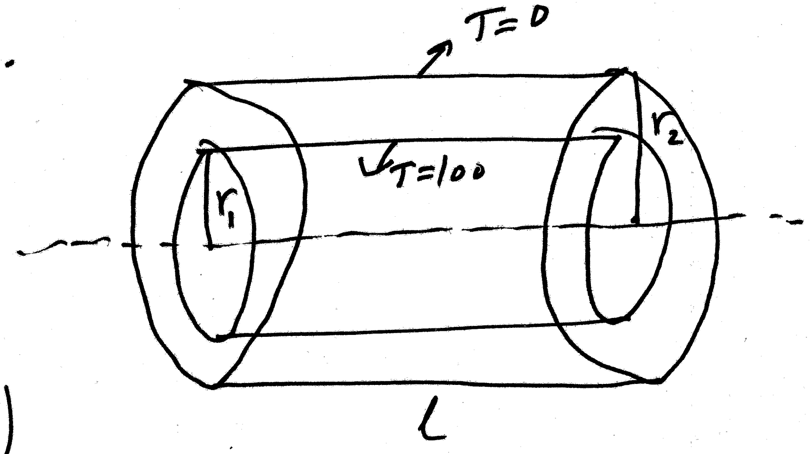
$\Rightarrow I(t) = I_0 e^{-\frac{R}{L}t}$

$= I_0 e^{-\frac{t}{(L/R)}} = I_0 e^{-t/\tau}$, with

$\tau = L/R$

time constant

(4) problem 8.2.23: Heat is escaping at a constant rate $\left[\frac{dQ}{dt} = B = \text{constant}\right]$ through the walls of long cylindrical pipe of inner radius $r_1 = 1$ and outer radius $r_2 = 2$. Find $T$ at a distance $r$ from the axis of the cylinder if $T(r_1) = 100$ and $T(r_2) = 0$.




Now $\frac{dQ}{dt} = kA\frac{dT}{dr} = B$

k: thermal conductivity

A: lateral area $(= 2\pi r l)$

$\Rightarrow \frac{dQ}{dt} = k(2\pi r l)\frac{dT}{dr} = B \Rightarrow \frac{dT}{dr} = \frac{B}{k2\pi l}\frac{1}{r}$

$\frac{B}{k2\pi l}$ constant $= C$

$\Rightarrow \frac{dT}{dr} = \frac{C}{r} \Rightarrow dT = C\frac{dr}{r}$, integrate

$\boxed{T = C\ln r + A}$; A constant of integration

to find C and A use $T(r_1) = 100$ and $T(r_2) = 0$

$T(1) = 100$, $T(2) = 0$

$\Rightarrow 100 = C\ln(1) + A \Rightarrow A = 100$, and

$0 = C\ln 2 + 100 \Rightarrow C = -\frac{100}{\ln 2}$

$\Rightarrow T(r) = -100\frac{\ln r}{\ln 2} + 100$

$T(r) = 100\left(1 - \frac{\ln r}{\ln 2}\right)$

⑤ problem 8.3.3: solve $dy + (2xy - xe^{-x^2})dx = 0$

$\Rightarrow \frac{dy}{dx} + 2xy = xe^{-x^2} \Rightarrow y' + 2xy = xe^{-x^2}$ ---(1)

$\mu = e^{\int 2x\,dx} = e^{\frac{2x^2}{2}} = e^{x^2}$, multiply (1) by $e^{x^2}$

$\Rightarrow e^{x^2}y' + 2xy e^{x^2} = x \Rightarrow [e^{x^2}y]' = x$, integrate

$e^{x^2}y = \frac{x^2}{2} + A \Rightarrow \boxed{y(x) = \left(\frac{1}{2}x^2 + A\right)e^{-x^2}}$

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⑥ problem 8.3.8: solve $\frac{dx}{dy} = \cos y - x\tan y$ ; Note

here that $\frac{dx}{dy} = x' \Rightarrow x' = \cos y - x\tan y$

$\Rightarrow \boxed{x' + \underbrace{\tan y}_{P(y)}\, x = \cos y}$ ---(2), $\mu = e^{\int \tan y\, dy}$

$= e^{-\ln\cos y} = e^{\ln\frac{1}{\cos y}}$

$= \frac{1}{\cos y} = \sec y$

multiply (2) by $\mu$, we get

$\{\because \boxed{\mu = \sec y}$

$x'\sec y + \sec y \tan y\, x = 1$

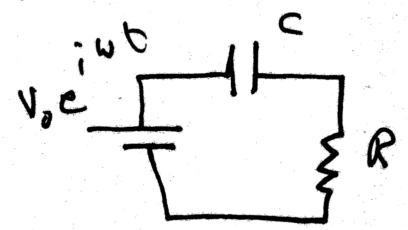
$[x\sec y]' = 1$ ; where $\frac{d}{dy}\sec y = \sec y\tan y$

integrate over $dy$

$x\sec y = y + A \Rightarrow \frac{x}{\cos y} = A + y$

$\Rightarrow \boxed{x = (A+y)\cos y}$

⑦ problem 8.3.18: solve equation (1.3) in textbook for $L=0$ and $V=V_0 e^{i\omega t}$




$$L^2 \frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = \frac{dV}{dt} \quad ;$$

$$R\frac{dI}{dt} + \frac{1}{C} I = V_0 i\omega e^{i\omega t} \,, \quad \text{divide by } R$$

$$\frac{dI}{dt} + \frac{1}{RC} I = \frac{V_0 i\omega}{R} e^{i\omega t} \Rightarrow \boxed{I' + \frac{1}{RC} I = \frac{V_0 i\omega}{R} e^{i\omega t}} \quad \text{--- (1)}$$

$$\mu = e^{\int \frac{1}{RC} dt} = e^{t/RC} \,, \quad \text{multiply (1) by } \mu$$

$$e^{t/RC} I' + \frac{1}{RC} I e^{t/RC} = \frac{V_0 i\omega}{R} e^{i\omega t} e^{t/RC}$$

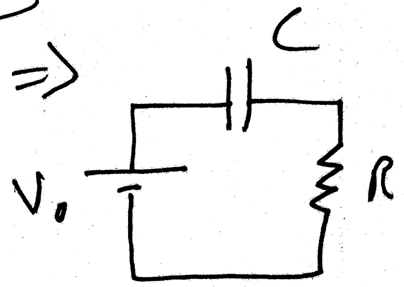
$$[I e^{t/RC}]' = \frac{V_0 i\omega}{R} e^{i\omega t} e^{t/RC} = \frac{V_0 i\omega}{R} e^{(i\omega + \frac{1}{RC})t} \quad \text{integrate over } dt$$

$$I e^{t/RC} = \frac{V_0 i\omega}{R} \frac{e^{(i\omega + \frac{1}{RC})t}}{(i\omega + \frac{1}{RC})} + A$$

$$\Rightarrow I = \frac{V_0 i\omega}{(i\omega R + \frac{1}{C})} e^{i\omega t} + A e^{-t/RC}$$

$$\boxed{I(t) = \frac{iV_0\omega C}{(i\omega RC + 1)} e^{i\omega t} + A e^{-t/RC}}$$

see that, when $\omega = 0$ (DC circuit) $\Rightarrow$




$I = A e^{-t/RC}$ ; at $t=0$, $I=I_0$ $\Rightarrow A = I_0$

$I = I_0 e^{-t/\tau}$ ; $\tau = RC$

⑧ Problem 8.4.2: solve $y' + \frac{1}{x} y = 2x^{3/2} y^{1/2}$ --- (1)

using Bernoulli method, let $z = y^{1-1/2} = y^{1/2}$, $z' = \underbrace{\frac{1}{2} y^{-1/2}} y'$.

Now multiply (1) by $\frac{1}{2} y^{-1/2}$, we get

$$\frac{1}{2} y^{-1/2} y' + \frac{1}{2} \frac{y y^{-1/2}}{x} = 2x^{3/2} y^{1/2} \cdot \frac{1}{2} y^{-1/2}$$

$$\frac{1}{2} y^{-1/2} y' + \frac{1}{2} \frac{y^{1/2}}{x} = x^{3/2} \Rightarrow z + \frac{z}{2x} = x^{3/2} \text{ ---- (2)}$$

$$\mu = e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2} \ln x} = e^{\ln x^{1/2}} = x^{1/2}, \text{ multiply (2) by } \mu$$

$$\Rightarrow x^{1/2} z + \frac{1}{2} z x^{-1/2} = x^2 \Rightarrow [x^{1/2} z]' = x^2, \text{ integrate}$$

$$\Rightarrow x^{1/2} z = \frac{x^3}{3} + A \Rightarrow z = \frac{1}{3} x^{5/2} + A x^{-1/2}, \text{ but } z = y^{1/2}$$

$$\Rightarrow \boxed{y^{1/2} = \frac{1}{3} x^{5/2} + A x^{-1/2}}$$

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⑨ Problem 8.4.9: solve $xy\,dx + (y^2 - x^2)\,dy = 0$ --- (1)

let $y = vx \Rightarrow dy = v\,dx + x\,dv$, substitute in (1), we get

$$x(vx)\,dx + (v^2x^2 - x^2)(v\,dx + x\,dv) = 0, \text{ simplify}$$

$$v^3 x^2\,dx + (v^2 - 1) x^3\,dv = 0 \Rightarrow \text{divide by } v^3 x^3$$

$$\frac{dx}{x} + \frac{v^2 - 1}{v^3} dv = 0 \Rightarrow \left(\frac{1}{v} - \frac{1}{v^3}\right) dv = -\frac{dx}{x}, \text{ integrate}$$

$$\ln v - \frac{v^{-2}}{-2} = -\ln x + \ln A \Rightarrow \ln v + \frac{1}{2v^2} = \ln\left(\frac{A}{x}\right)$$

$$\Rightarrow \ln \frac{v}{(A/x)} = -\frac{1}{2v^2} \Rightarrow \frac{v}{A/x} = e^{-\frac{1}{2v^2}} \Rightarrow \frac{vx}{A} = e^{-\frac{1}{2v^2}}$$

$$\text{but } y = vx \Rightarrow \frac{y}{A} = e^{-\frac{1}{2y^2/x^2}}$$

$$\Rightarrow \boxed{y = A e^{-\frac{x^2}{2y^2}}} \Rightarrow \text{or } \boxed{y^2 = A^2 e^{-\frac{x^2}{y^2}} \quad y^2 = C e^{-\frac{x^2}{y^2}}}$$