Mathematical physics (1)
HW # 8 – solution
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① problem 7.4.7 Find the average of $x-\cos^2 6x$ over $[0, \pi/6]$

$$\overline{x-\cos^2 6x} = \frac{1}{\frac{\pi}{6}-0}\int_0^{\pi/6} x-\cos^2 6x \, dx$$

$$= \frac{6}{\pi}\left[\int_0^{\pi/6} x\,dx - \int_0^{\pi/6}\cos^2 6x\,dx\right] \text{; using } \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{6}{\pi}\left\{\frac{x^2}{2}\Big|_0^{\pi/6} - \int_0^{\pi/6} \tfrac{1}{2}(1+\cos 12x)\,dx\right\}$$

$$= \frac{6}{\pi}\left[\frac{\pi^2}{72} - \frac{1}{2}\int_0^{\pi/6} dx - \underbrace{\frac{1}{2}\int_0^{\pi/6}\cos 12x\,dx}_{\text{Zero as period } \frac{2\pi}{12}=\frac{\pi}{6}}\right]$$

$$= \frac{6}{\pi}\left[\frac{\pi^2}{72} - \frac{1}{2}\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{6}{\pi}\left\{\frac{\pi^2}{72} - \frac{\pi}{12}\right\} = \frac{\pi}{12} - \frac{1}{2}$$

② find the following integrals without deep calculations

a) problem 7.4.14 : $\int_{-\pi/2}^{3\pi/2} \cos^2\left(\frac{x}{2}\right) dx$

the period of $\cos^2\left(\frac{x}{2}\right)$ is $\frac{2\pi}{1/2} = 4\pi$. Here we integrate over half the period of $\cos^2\left(\frac{x}{2}\right)$ which is $\frac{3\pi}{2} - \left(-\frac{\pi}{2}\right)$

$$= \frac{3\pi}{2} + \frac{\pi}{2} = \frac{4\pi}{2} = 2\pi$$

$\Rightarrow$ so

$$\int_{-\pi/2}^{3\pi/2} \cos^2\left(\frac{x}{2}\right) dx = \frac{1}{2}\int_{\text{full period}} \cos^2\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{2} \times \overline{\cos^2\left(\frac{x}{2}\right)} \times (4\pi - 0) = \frac{1}{2} \times \frac{1}{2} \times 4\pi = \pi$$

b) problem 7.4.16(a) $\int_0^{\frac{2\pi}{w}} \sin^2 wt \, dt$

Period of $\sin^2 wt$ is $\frac{2\pi}{w}$

$$\Rightarrow \int_0^{\frac{2\pi}{w}} \sin^2 wt \, dt = \overline{\sin^2 wt} \times \left(\frac{2\pi}{w} - 0\right) = \frac{1}{2} \times \frac{2\pi}{w} = \frac{\pi}{w}$$

c) problem 7.4.16 (b), $\int_0^2 \cos^2 2\pi t \, dt$

the period of $\cos^2 2\pi t$ is $\frac{2\pi}{2\pi} = 1$, so we integrate here over twice the period

$$\Rightarrow \int_0^2 \cos^2 2\pi t \, dt = 2 \times \underbrace{\int \cos^2 2\pi t \, dt}_{\text{full period}} = 2 \times \overline{\cos^2 2\pi t} \times (1-0)$$

$$= 2 \times \frac{1}{2} \times 1 = 1$$

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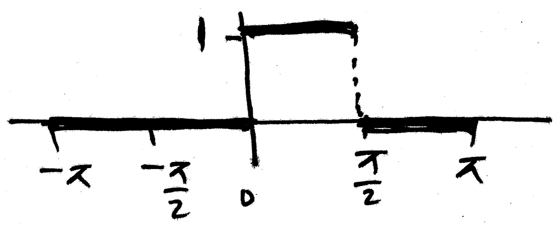
③ problem 7.5.2: expand the following function in sine-cosine Fourier series and sketch the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$



$[-\pi, \pi]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi/2} (1)\,dx = \frac{1}{\pi} x \Big|_0^{\pi/2} = 1/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi/2} \cos nx\,dx = \frac{1}{\pi}\frac{\sin nx}{n}\Big|_0^{\pi/2}$$

$$= \frac{1}{\pi}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi/2} \sin nx\,dx = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi/2}$$

$$= -\frac{1}{\pi n}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] = \frac{1}{\pi}\,\frac{\left[1-\cos\left(\frac{n\pi}{2}\right)\right]}{n}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos nx + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-\cos\frac{n\pi}{2}\right)}{n}\sin nx$$

$$= \frac{1}{4} + \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right)$$

$$+ \frac{1}{\pi}\left(\frac{\sin x}{1} + \sin 2x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

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(4) problem 7.5.8: expand $f(x) = 1+x$ over $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\,dx = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\cos nx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi}\cos nx\,dx}_{0} + \underbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}_{0}\right] = 0$$

odd function

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi}\sin nx\,dx}_{\text{zero}} + \underbrace{\int_{-\pi}^{\pi} x\sin nx\,dx}_{-\frac{2\pi}{n}(-1)^n}\right]$$

by parts, let
$u = x$, $dv = \sin nx\,dx$
$du = dx$, $v = -\frac{\cos nx}{n}$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n\right] = \frac{-2(-1)^n}{n}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx = 1 - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= 1 - 2\left(-\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \frac{\sin 4x}{4} - \cdots\right)$$

$$= 1 + 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

⑤ expand $g(x) = -3-3x$ over $[-\pi, \pi]$

using the result of last problem, we have

$$g(x) = -3-3x = -3(1+x) = -3f(x)$$

$$= -3\left[1 + 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)\right]$$

$$= -3 - 6\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

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⑥ problem 7.7.1 expand the following function in Fourier series of complex exponentials e^{inx} over $[-\pi, \pi]$

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{with } C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}$$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} dx = \frac{1}{2\pi} x\Big|_{-\pi}^{0} = \frac{1}{2\pi}[0-(-\pi)]$$

$$= 1/2$$

and

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx} = -\frac{1}{2\pi i n}\left[e^{-inx}\right]_{-\pi}^{0} = -\frac{1}{2\pi i n}\left[1 - e^{i\pi n}\right]$$

$$= \frac{1}{2\pi i n}\left[e^{i\pi n} - 1\right] = \frac{1}{2\pi i n}\left\{\cos n\pi + i\sin n\pi - 1\right\}$$

$$= \frac{1}{2\pi i n}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{i}{\pi n}, & n \text{ odd} \end{cases}$$

$$\Rightarrow f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ \text{odd } n,\ n\neq 0}}^{\infty} \frac{(-1)^n - 1}{2\pi i n} e^{inx}$$

$$= \frac{1}{2} + \frac{1}{2\pi i}\sum_{\substack{n=-\infty \\ \text{odd } n,\ n\neq 0}}^{\infty} \frac{(-1)^n - 1}{n} e^{inx}$$

$f(x)$ can also be written in sine-cosine series as follows

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

now using $C_0 = 1/2 = \frac{a_0}{2}$ and $C_n = \frac{c'}{\pi n}$, odd n

$$\boxed{C_{-n} = C_n^* = -\frac{c'}{\pi n}}$$

$$\Rightarrow a_n = C_n + C_{-n} = \frac{c'}{\pi n} - \frac{c'}{\pi n} = 0 \text{ , and}$$

$$b_n = c'(C_n - C_{-n}) = c'\left(\frac{c'}{\pi n} + \frac{c'}{\pi n}\right) = -\left(\frac{2}{n\pi}\right)$$

$$\Rightarrow f(x) = \frac{1}{2} - \frac{2}{\pi} \sum_{\substack{n=1 \\ \text{odd } n \\ n \neq 0}}^{\infty} \frac{\sin nx}{n}$$

$$= \frac{1}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{3} + \cdots \right)$$

which is identical to result of problem 7.7.1 obtained before.

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7 problem 7.8.13 (a) $f(x) = 2 - x$; $-2 < x < 2$ $\Rightarrow l = 2$

expand in Fourier series of complex exponentials.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{2}} \quad ; \quad C_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx$$

$$\Rightarrow C_0 = \frac{1}{4} \int_{-2}^{2} (2-x)\, dx = \frac{1}{4} \left[2x - \frac{x^2}{2}\right]_{-2}^{2} = 2 \text{ , and}$$

$$C_n = \frac{1}{4} \int_{-2}^{2} (2-x) e^{-\frac{in\pi x}{2}} dx$$

$$= \frac{1}{4} \left[\underbrace{\int_{-2}^{2} 2 e^{-\frac{in\pi x}{2}} dx}_{\mathrm{I}} - \underbrace{\int_{-2}^{2} x e^{-\frac{in\pi x}{2}} dx}_{\mathrm{II}} \right]$$

$$I = 2\int_{-2}^{2} e^{-\frac{in\pi x}{2}} dx = \frac{-4}{in\pi} e^{-\frac{in\pi x}{2}}\Big|_{-2}^{2} = \frac{-4}{i\pi n}\left[e^{-in\pi} - e^{in\pi}\right]$$

$$= \frac{4}{i\pi n}\left(e^{in\pi} - e^{-in\pi}\right)$$

and

$$II = \int_{-2}^{2} x e^{-\frac{in\pi x}{2}} dx$$

integrate by parts let $u = x$, $du = dx$, $dv = e^{-\frac{in\pi x}{2}} dx$

$$v = \frac{-2}{in\pi} e^{-\frac{in\pi x}{2}}$$

$$II = -\frac{2x}{in\pi} e^{-\frac{in\pi x}{2}}\Big|_{-2}^{2} + \frac{2}{in\pi}\int_{-2}^{2} e^{-\frac{in\pi x}{2}}$$

$$= -\frac{4}{in\pi}\left(e^{in\pi} + e^{-in\pi}\right) + \frac{4}{n^2\pi^2}\left(e^{-in\pi} - e^{in\pi}\right)$$

$$\Rightarrow C_n = \frac{1}{4}\left[\frac{4}{i\pi n}\left(e^{in\pi} - e^{-in\pi}\right) + \frac{4}{in\pi}\left(e^{in\pi} + e^{-in\pi}\right) - \frac{4}{n^2\pi^2}\left(e^{-in\pi} - e^{in\pi}\right)\right]$$

$$= \frac{e^{in\pi}}{i\pi n} - \cancel{\frac{e^{-in\pi}}{i\pi n}} + \frac{e^{in\pi}}{i\pi n} + \cancel{\frac{e^{-in\pi}}{i\pi n}} + \frac{4}{n^2\pi^2}\left(e^{in\pi} - e^{-in\pi}\right)$$

$$= \frac{2}{i\pi n} e^{in\pi} + \frac{4}{n^2\pi^2}\left(e^{in\pi} - e^{-in\pi}\right)$$

$$= \frac{2}{i\pi n}\left(\cos n\pi + i\underbrace{\sin n\pi}_{0 \to \text{as } n \text{ is integer}}\right) + \frac{4}{n^2\pi^2}\left[\cancel{\cos n\pi} + i\underbrace{\sin n\pi}_{0} - \cancel{\cos n\pi} + i\underbrace{\sin n\pi}_{0}\right]$$

$$= \frac{2}{i\pi n}\cos n\pi = \frac{2}{i\pi n}(-1)^n$$

$$\Rightarrow f(x) = 2 + \frac{2}{i\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{\frac{in\pi x}{2}}$$ ✓

Now to find Fourier series in sines and cosines we proceed as follow

$$c_0 = \frac{a_0}{2} = 2, \quad c_n = \frac{2}{i\pi}\frac{(-1)^n}{n}, \quad c_{-n} = c_n^* = -\frac{2}{i\pi}\frac{(-1)^n}{n}$$

$$\Rightarrow a_n = c_n + c_{-n} = 0$$

$$b_n = i(c_n - c_{-n}) = i\left(\frac{2}{i\pi}\frac{(-1)^n}{n} + \frac{2}{i\pi}\frac{(-1)^n}{n}\right)$$

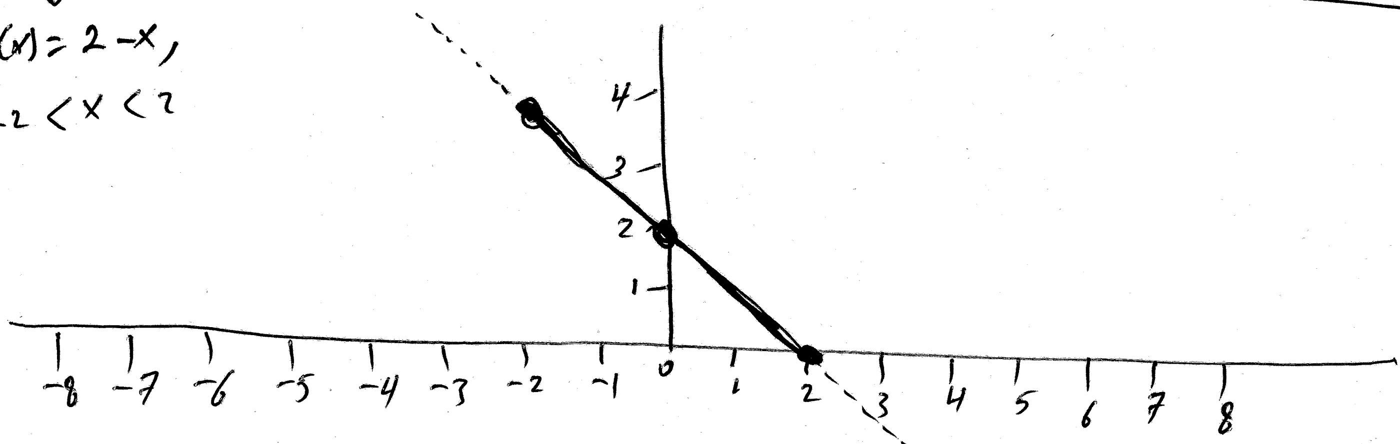
$$= \frac{4}{\pi}\frac{(-1)^n}{n}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{a_n}_{0} \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$= 2 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)$$

$f(x) = 2 - x,$

$-2 < x < 2$



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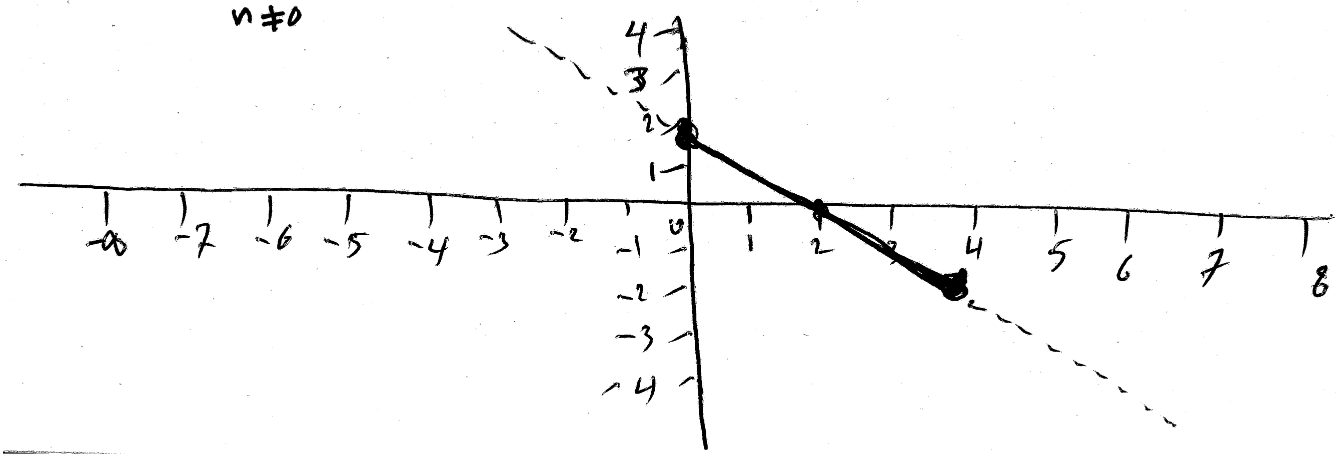
in Part (b), $f(x) = 2 - x$, $0 < x < 4$. here interval is changed, so we have different function, therefore Fourier series are expected to be different

$$f(x) = \frac{2}{i\pi}\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{i\frac{n\pi x}{2}} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

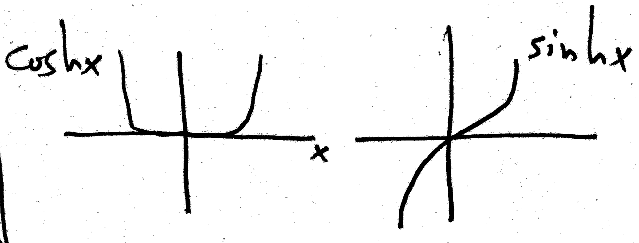


⑧ problems 7.9.(1–3): write the following functions as the sum of an even function and odd function

9.1 (a) $e^{inx} = \underbrace{\cos nx}_{\text{even}} + i\underbrace{\sin nx}_{\text{odd}}$;

Note: even × even = even
odd × odd = even
odd × even = odd

9.1 (b) $xe^{x} = x[\cosh x + \sinh x] = \underbrace{x\cosh x}_{\text{odd}} + \underbrace{x\sinh x}_{\text{even}}$

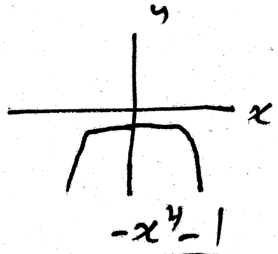


9.2 (b)

$(1+x)(\sin x + \cos x) = \sin x + \cos x + x\sin x + x\cos x$

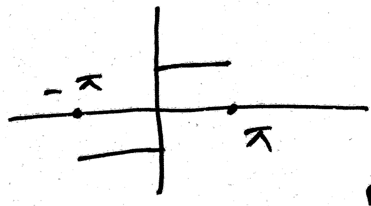
$= \underbrace{(\cos x + x\sin x)}_{\text{even}} + \underbrace{(\sin x + x\cos x)}_{\text{odd}}$

9.3 (a) $x^5 - x^4 + x^3 - 1 = \underbrace{(-x^4 - 1)}_{\text{even}} + \underbrace{(x^5 + x^3)}_{\text{odd}}$



⑨ problem 7.9.5: expand the following function in Fourier series

$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$; $[-\pi, \pi]$ $l = \pi$



odd function

$\Rightarrow a_0 = 0$

$a_n = 0$

$\Rightarrow b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$; $l = \pi$

$= \frac{2}{\pi}\int_0^\pi \sin nx\,dx = -\frac{2}{n\pi}[\cos n\pi - 1]$

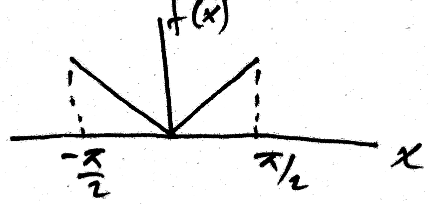
$= \frac{2}{n\pi}[1 - \cos(n\pi)] = \frac{2}{n\pi}[1 - (-1)^n] = \begin{cases} 0, & \text{even } n \\ \frac{4}{n\pi}, & \text{odd } n \end{cases}$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{4}{\pi}\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n}\sin nx$

$= \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$

(10) problem 7.9.10: expand the following function in Fourier series

$$f(x) = |x| = \begin{cases} -x, & -\frac{\pi}{2} < x < 0 \\ x, & 0 < x < \frac{\pi}{2} \end{cases};$$



$\because [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow l = \pi/2$

$f(x)$ is even $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi/2}\int_0^{\pi/2} x\,dx = \frac{4}{\pi}\frac{x^2}{2}\Big|_0^{\pi/2} = \frac{\pi}{2} \Rightarrow \frac{a_0}{2} = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} x\cos\frac{n\pi x}{l}\,dx\ ;\ l = \pi/2$$

$$= \frac{4}{\pi}\int_0^{\pi/2} x\cos 2nx\,dx$$; integrate by parts, let $u = x$, $dv = \cos 2nx\,dx$; $du = dx$, $v = \frac{\sin 2nx}{2n}$

$$= \frac{4}{\pi}\left[\frac{x\sin 2nx}{2n}\Big|_0^{\pi/2} - \frac{1}{2n}\int_0^{\pi/2}\sin 2nx\,dx\right]$$

$$= \frac{4}{\pi}\left[\frac{\pi}{4n}\underbrace{\sin n\pi}_{0} - \frac{1}{2n}\left[-\frac{\cos 2nx}{2n}\right]_0^{\pi/2}\right]$$

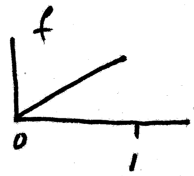
$$= \frac{4}{\pi}\cdot\frac{1}{4n^2}\left[\cos n\pi - \cos 0\right] = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0, & \text{even } n \\ -\frac{2}{\pi n^2}, & \text{odd } n \end{cases}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos 2nx$$

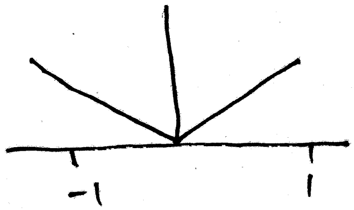
$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty}\frac{1}{n^2}\cos 2nx$$

(11) <u>problem 7.9.15</u> given $f(x) = x$; $0 < x < 1$



a) expand $f(x)$ in Fourier cosine series

$\Rightarrow$ extend $f(x)$ to $(-1, 0)$ and make it even with period $2 = 2\ell \Rightarrow \ell = 1$



$f(x)$ is now even $\Rightarrow$ $b_n = 0$

$$a_0 = \frac{2}{\ell}\int_0^\ell f(x)\,dx = 2\int_0^1 x\,dx = 1$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{even } n \\ -\frac{4}{n^2\pi^2}, & \text{odd } n \end{cases}$$

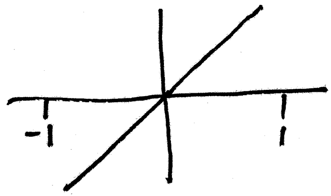
$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} \; ; \; \ell = 1$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2}\cos n\pi x$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos\pi x}{1} + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \cdots\right.$$

b) expand $f(x)$ in Fourier sine series $\Rightarrow$

extend $f(x)$ to $(-1, 0)$ and make it odd with period $2 = 2\ell \Rightarrow \ell = 1$



$f(x)$ is now odd $\Rightarrow$ $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\int_0^1 x\sin(n\pi x)\,dx = -\frac{2}{\pi n}(-1)^n = \frac{2}{\pi n}(-1)^{n+1}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$$

$$= \frac{2}{\pi}\left[\frac{\sin\pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right]$$

(12) problem 7.11.5: using Parsevals' theorem applied to problem 7.9.6, Find the sum of the series $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

$$f(x)=\begin{cases}-1, & -l<x<0\\ 1, & 0<x<l\end{cases} \Rightarrow f(x)=\frac{4}{\pi}\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

see that $a_0=0$, $a_n=0$, and

$b_n=\frac{4}{\pi n}$; where n is odd, now using Parsevals' theo

$$\frac{1}{2l}\int_{-l}^{l}[F(x)]^2dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}a_n^2+b_n^2=\frac{1}{2}\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty}b_n^2$$

$$\frac{1}{2l}\int_{-l}^{0}(-1)^2dx+\frac{1}{2l}\int_{0}^{l}(1)^2dx=\frac{1}{2}\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty}\frac{16}{\pi^2n^2}=\frac{8}{\pi^2}\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{2l}\{l+l\}=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow 1=\frac{8}{\pi^2}\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

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(13) problem 7.11.6: Find $\sum_{n=1}^{\infty}\frac{1}{n^4}$ from problem 7.9.9

$$f(x)=x^2,\ -\tfrac{1}{2}<x<\tfrac{1}{2}\Rightarrow f(x)=\frac{1}{12}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 2\pi n x$$

$\Rightarrow a_0=\frac{1}{6}$, $a_n=\frac{(-1)^n}{\pi^2n^2}$; $b_n=0$, using Parsevals' theorem with $l=\frac{1}{2}$

$$\Rightarrow\frac{1}{2\times\frac{1}{2}}\int_{-1/2}^{1/2}F(x)^2dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}a_n^2+\cancel{b_n^2}^{\to 0}$$

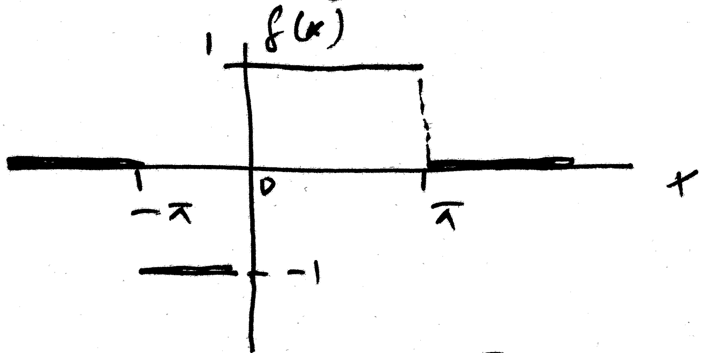
$$\int_{-1/2}^{1/2}x^4dx=\frac{1}{36\times4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\pi^4n^4}\Rightarrow\frac{1}{80}=\frac{1}{144}+\frac{1}{2\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^4}=2\pi^4\left(\frac{1}{80}-\frac{1}{144}\right)=$$

$$=2\pi^4\frac{64}{11520}=\frac{\pi^4}{(11520/120)}=\frac{\pi^4}{90}\equiv\zeta(4)$$

(14) problem 7.12.3: write the following function as a Fourier integral

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$



$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -e^{-ikx}\, dx + \int_{0}^{\pi} e^{-ikx}\, dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{e^{-ikx}}{-ik}\Big|_{-\pi}^{0} + \frac{e^{-ikx}}{-ik}\Big|_{0}^{\pi}\right] = \frac{1}{2\pi i k}\left[e^{-ikx}\Big|_{-\pi}^{0} - e^{-ikx}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi i k}\left[(1 - e^{ik\pi}) - (e^{-ik\pi} - 1)\right]$$

$$= \frac{1}{2\pi i k}\{1 - e^{ik\pi} - e^{-ik\pi} + 1\} = \frac{1}{2\pi i k}\left[2 - (e^{ik\pi} + e^{-ik\pi})\right]$$

$$= \frac{1}{2\pi i k}\left[2 - 2\cos k\pi\right] = \frac{1}{\pi i k}\left[1 - \cos k\pi\right]$$

$$\Rightarrow f(x) = \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk$$

$$= \int_{-\infty}^{\infty} \frac{1 - \cos k\pi}{ik\pi}\, e^{ikx}\, dk$$

(15) <u>problem 7.12.23:</u> using the result of last problem show that

$$\int_0^\infty \frac{1-\cos\pi k}{k} \sin k \, dk = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{1-\cos\pi k}{k} \sin \pi k \, dk = \frac{\pi}{4}$$

Now from last problem, we have

$$f(x) = \int_{-\infty}^{\infty} \frac{1-\cos k\pi}{ik\pi} e^{ikx} dk = \int_{-\infty}^{\infty} \frac{1-\cos\pi k}{ik\pi} [\cos kx + i \sin kx] dk$$

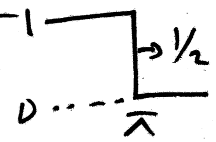
$$= \frac{1}{i\pi} \underbrace{\int_{-\infty}^{\infty} \underbrace{\underbrace{\frac{1-\cos\pi k}{k}}_{\text{odd}} \underbrace{\cos kx}_{\text{even}}}_{\text{odd}} dk}_{= \text{zero}} + \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underbrace{\frac{1-\cos\pi k}{k}}_{\text{odd}} \underbrace{\sin kx}_{\text{odd}}}_{\text{even}} dk$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1-\cos\pi k}{k} \sin kx \, dk = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & \pi < x < -\pi \end{cases}$$

i) let $x = 1$ $\Rightarrow \frac{2}{\pi} \int_0^\infty \frac{1-\cos\pi k}{k} \sin k \, dk = 1$

$$\Rightarrow \int_0^\infty \frac{1-\cos\pi k}{k} \sin k \, dk = \pi/2$$

ii) let $x = \pi$, here we take $f(x) = f(\pi) = 1/2$ as the fourier integral at the jump is defined by the midpoint



$$\Rightarrow \frac{2}{\pi} \int_0^\pi \frac{1-\cos\pi k}{k} \sin \pi k \, dk = 1/2$$

$$\Rightarrow \int_0^\pi \frac{1-\cos\pi k}{k} \sin \pi k \, dk = \frac{\pi}{4}$$