Mathematical physics (1)
HW #7 – Solution
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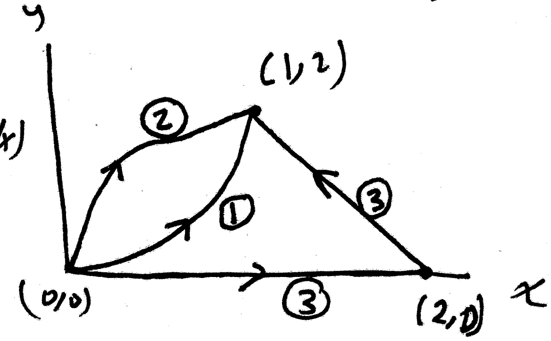
① problem 6.8.1 : evaluate the line integral $\int (x^2-y^2)dx - 2xy\,dy$ along the following paths from $(0,0)$ to $(1,2)$

a) along path (1) $y = 2x^2$, $dy = 4x\,dx$



$$\int (x^2-y^2)dx - 2xy\,dy = \int_0^1 (x^2-4x^4)dx - 2x(2x^2)(4x\,dx)$$

$$= \int_0^1 x^2 - 4x^4 - 16x^4 = \int_0^1 (x^2 - 20x^4)\,dx$$

$$= -\frac{11}{3}$$

b) along path (2) ; $x = t^2$, $dx = 2t\,dt$, $y = 2t$, $dy = dt$ $\Rightarrow$ $(0,0) \to t=0$, $(1,2) \to t=1$

$$\int (x^2-y^2)dx - 2xy\,dy = \int_0^1 (t^4 - 4t^2)\,2t\,dt - 2t^2(2t)(2dt)$$

$$= \int_0^1 2t^5 - 8t^3 - 8t^3 = \int_0^1 2t^5 - 16t^3 = -\frac{11}{3}$$

c) along $y=0$ from $(0,0) \to (2,1)$ and then along line from $(2,0) \to (1,2)$.

line connecting $(2,0) \to (1,2)$ $\Rightarrow y = ax+b$; $a = \text{slope} = \frac{2-0}{1-2} = -2$

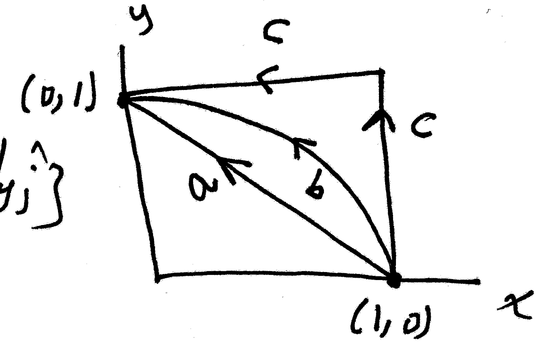
$\Rightarrow y = -2x + b$. to find b substitute the point $(2,0)$

$\Rightarrow 0 = -2(2) + b \Rightarrow b = 4$ $\Rightarrow \boxed{y = -2x+4}$ $\Rightarrow dy = -2dx$

$$\int_{(0,0)}^{(2,0)} (x^2-\underbrace{y^2}_{0})dx - \underbrace{2xy}_{0}\underbrace{dy}_{0} + \int_{(2,0)}^{(1,2)} (x^2 - (-2x+4)^2)dx - 2x(-2x+4)(-2dx)$$

$$\int_0^2 x^2 dx + \int_2^1 (-11x^2 + 32x - 16)\,dx = \frac{8}{3} - \frac{19}{3} = -\frac{11}{3}$$

② **problem 6.8.6:** Find the work done by the force $\vec{F} = (2xy-3)\hat{i} + x^2\hat{j}$ in moving an object from $(1,0)$ to $(0,1)$ along the following paths.



a) along the straight line (a)

$$W_a = \int \vec{F}\cdot d\vec{r} = \int [(2xy-3)\hat{i} + x^2\hat{j}]\cdot[dx\hat{i} + dy\hat{j}]$$

$$= \int_a (2xy-3)\,dx + \int_a x^2\,dy$$

the eqn of line is $y = ax+b$; $a = \text{slope} = \frac{1-0}{0-1} = -1 \Rightarrow y = -x+b$

to find b, substitute the point $(1,0) \Rightarrow 0 = -1+b \Rightarrow b = 1$

$\Rightarrow y = -x+1 \Rightarrow dy = -dx$

$$\Rightarrow W_a = \int_1^0 (2x(-x+1)-3)\,dx + \int_1^0 x^2(-dx) = \int_1^0 (-3x^2+2x-3)\,dx = +3$$

b) along the circular path $x^2+y^2 = 1 \Rightarrow y^2 = 1-x^2$

$\Rightarrow 2y\,dy = -2x\,dx \Rightarrow dy = -\frac{x}{y}dx = -\frac{x}{\sqrt{1-x^2}}dx$

$$W_b = \int_0^1 \left[2x(1-x^2)^{1/2} - 3 \frac{-x^3}{\sqrt{1-x^2}}\right]dx = +3$$

it can also be done in polar coordinates $x = r\cos\theta = \cos\theta$; where $r=1$
$y = r\sin\theta = \sin\theta$

$\Rightarrow dx = -\sin\theta\,d\theta,\ dy = \cos\theta\,d\theta$

$$W_b = \int (2\cos\theta\sin\theta - 3)(-\sin\theta\,d\theta) + \int \cos^2\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} [-2\cos\theta\sin^2\theta + 3\sin\theta + \cos^3\theta]\,d\theta = +3$$

c) from $(1,0) \to (1,1) \to (0,1)$

$$W_c = \int_{(1,0)}^{(1,1)} [(2xy-3)\underbrace{dx}_{0} + x^2dy] + \int_{(1,1)}^{(0,1)} [(2xy-3)dx + \underbrace{x^2dy}_{0}]$$

$$= \int_{0,\ x=1}^{1,\ x=1} x^2dy + \int_{1,\ y=1}^{0,\ y=1} (2xy-3)\,dx = \int_0^1 dy + \int_1^0 (2x-3)\,dx = 1+2 = 3$$

(3) verify that the following force is conservative. if yes, find the Potential ϕ. $\vec{F} = \hat{i} - z\hat{j} - y\hat{k} = (1, -z, -y)$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & -z & -y \end{vmatrix} = 0 \Rightarrow \vec{F} \text{ is conservative}$$

$$\Rightarrow \vec{F} = -\nabla \phi$$

$$\phi = -\int_A^B \vec{F} \cdot d\vec{r} = -\int_A^B (1, -z, -y) \cdot (dx, dy, dz) = -\int_A^B dx - z\,dy - y\,dz$$

taking the reference point A at the origin $A(0,0,0)$ and point B as (x, y, z), we get

$$\phi = -\int_{(0,0,0)}^{(x,y,z)} dx - z\,dy - y\,dz \text{ ; the path integral can be}$$

taken as $\underbrace{(0,0,0) \to (x,0,0)}_{\substack{y=z=0\\ dy=dz=0}} \to \underbrace{(x,y,0)}_{\substack{z=0,\ x=\text{const}\\ dx=dz=0}} \to \underbrace{(x,y,z)}_{\substack{x=\text{const},\ y=\text{const}\\ dx=dy=0}}$

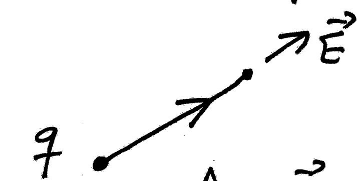
$$\Rightarrow \phi = -\int_{\substack{(0,0,0)\\ y=z=0\\ dy=dz=0}}^{(x,0,0)} dx - z\,dy - y\,dz - \int_{\substack{(x,0,0)\\ z=0\\ dx=dz=0}}^{(x,y,0)} dx - z\,dy - y\,dz - \int_{\substack{(x,y,0)\\ dx=dy=0\\ y=\text{const}}}^{(x,y,z)} dx - z\,dy - y\,dz$$

$$= -\int_0^x dx + y\int_0^z dz = -x + yz = yz - x$$

$$\therefore \phi(x, y, z) = yz - x$$

see that $\phi(0,0,0) = 0$ consistant with our choice of having the zero point potential is at the origin

(4) problem 6.8.19: show that the electric field of a point charge is conservative and find the corresponding potential



$$\hat{r} = \frac{\vec{r}}{r}$$

$$\vec{E} = k_e \frac{q}{r^2}\hat{r} \;; \text{ where } \hat{r} = \frac{\vec{r}}{r}$$

$$= k_e \frac{q}{r^3}\vec{r} \Rightarrow \nabla \times \vec{E} = \nabla \times \left(k_e \frac{q}{r^2}\vec{r}\right)$$

$$= k_e q \, \nabla \times \frac{\vec{r}}{r^2}$$

using $\nabla \times (\phi \vec{v}) = \phi \nabla \times \vec{v} + (\nabla\phi) \times \vec{v}$ with $\phi = \frac{1}{r^2}$, $\vec{v} = \vec{r}$

$$= k_e q \left[\frac{1}{r^2} \underbrace{\nabla \times \vec{r}}_{\text{zero}} + \underbrace{\nabla\left(\frac{1}{r^2}\right)}_{\frac{-2}{r^4}\vec{r}} \times \vec{r} \right]$$

$$= k_e q \left[0 - \frac{2}{r^4} \underbrace{\vec{r} \times \vec{r}}_{\text{zero}} \right] = \text{zero}$$

$\Rightarrow \vec{E}$ is conservative

$$\phi = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = \int_{r}^{\infty} \vec{E} \cdot d\vec{r} \;; \quad d\vec{r} = dr\,\hat{r}$$

$$= \int_{r}^{\infty} k_e \frac{q}{r^2}\hat{r} \cdot dr\,\hat{r} = k_e q \int_{r}^{\infty} \frac{dr}{r^2} = k_e q \int_{r}^{\infty} r^{-2}\,dr$$

$$= k_e q \left. \frac{r^{-1}}{-1} \right|_{r}^{\infty} = -k_e q \left[\frac{1}{r}\right]_{r}^{\infty} = -k_e q \left[\frac{1}{\infty} - \frac{1}{r}\right]$$

$$= k_e \frac{q}{r} \text{ as expected}$$

⑤ problem 6.8. 20 !

a) show that the gravitational force near the surface of the earth ($\vec{F} = -mg\,\hat{k}$) is conservative and find the corresponding potential

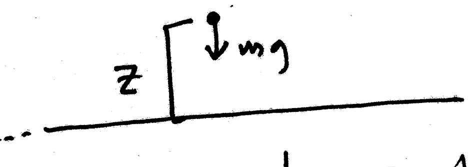
Note that here $\vec{F}$ is constant near the surface of earth

$$\vec{F} = -mg\,\hat{k} \Rightarrow \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & -mg \end{vmatrix} = \text{zero}$$

$\Rightarrow \vec{F}$ is conservative

$$\phi = -\int_0^z \vec{F}\cdot d\vec{r}$$

$$= -\int_0^z -mg\,dz = mgz \checkmark$$



b) if $\vec{F}$ is varying with distance from the center as $\vec{F} = -\dfrac{c}{r^3}\vec{r}$; where c is constant

show that $\vec{F}$ is also conservative

$$\nabla \times \vec{F} = \nabla \times \left(\frac{-c}{r^3}\vec{r}\right) = -c\,\nabla \times \frac{\vec{r}}{r^3}$$

$$= -c\left[\frac{1}{r^3}\underbrace{\nabla \times \vec{r}}_{0} + \underbrace{\nabla\left(\frac{1}{r^3}\right)}_{-\frac{3}{r^5}\vec{r}} \times \vec{r}\right] = +c\,\frac{3}{r^5}\,\vec{r}\times\vec{r}$$

$$= \text{zero}$$

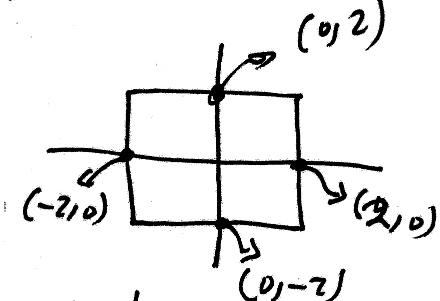
$$\phi = -\int_\infty^r \vec{F}\cdot d\vec{r} = -\int_\infty^r -\frac{c}{r^3}\vec{r}\cdot d\vec{r} = c\int_\infty^r \frac{\vec{r}}{r^3}\cdot d\vec{r} \; ; \quad \vec{r} = r\hat{r}, \; d\vec{r} = dr\,\hat{r}$$

$$= c\int_\infty^r \frac{r\,dr}{r^3} = c\int_\infty^r \frac{dr}{r^2} = c\,\frac{r^{-1}}{-1}\Big|_\infty^r = -c\left[\frac{1}{r}\right]_\infty^r$$

$$= -c\left[\frac{1}{r} - \frac{1}{\infty}\right] = -\frac{c}{r} \; ; \quad \phi = 0 \text{ at } \infty \text{ reference}$$

⑥ problem 6.9.2: evaluate $\oint 2x\,dy - 3y\,dx$ around the square with vertices $(0,2), (2,0), (-2,0), (0,-2)$.

- using Green's theorem in plane



$$\oint_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

with $P = -3y$ and $Q = 2x$

$$\oint_C 2x\,dy - 3y\,dx = \iint_A (2-(-3))\,dx\,dy = 5\iint dx\,dy$$

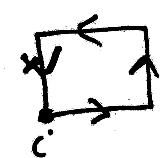
$$= 5\int_{-2}^{2} dx \int_{-2}^{2} dy = 5(2+2)(2+2) = 80$$

we can do the same integral around the closed path

$\oint_C \vec{F}\cdot d\vec{r}$; with $\vec{F} = -3y\,\hat{i} + 2x\,\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

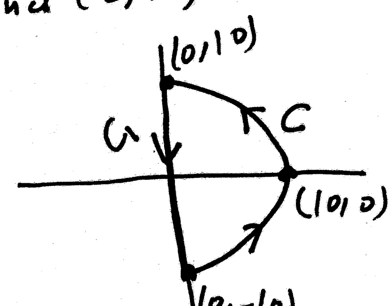
$$\oint_C \vec{F}\cdot d\vec{r} = \oint(-3y\hat{i}+2x\hat{j})\cdot(dx\hat{i}+dy\hat{j}) = \oint -3y\,dx + 2x\,dy$$

$$= \underset{y=-2}{\int_{-2}^{2} 6\,dx} + \underset{x=2}{\int_{-2}^{2} 4\,dy} + \underset{y=2}{\int_{2}^{-2} -6\,dx} + \underset{x=-2}{\int_{2}^{-2} -4\,dx} = 80$$



⑦ problem 6.9.5: evaluate $\int_C (ye^x - 1)\,dx + e^x\,dy$, where C is the semicircle through $(0,-10)$, $(10,0)$ and $(0,10)$

using Green's theorem, we have



$$\oint_{C+C_1} (ye^x - 1)\,dx + e^x\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

with $P = ye^x - 1$ and $Q = e^x$

$$\frac{\partial P}{\partial y} = e^x, \quad \frac{\partial Q}{\partial x} = e^x$$

$$\Rightarrow \oint_{C+C_1} (ye^x - 1)\,dx + e^x\,dy = \iint_A (e^x - e^x)\,dx\,dy = \text{zero}$$

$$\Rightarrow \int (ye^x - 1)\,dx + e^x\,dy + \int (ye^x - 1)\,dx + e^x\,dy = 0$$

$$\Rightarrow \int_C (ye^x - 1)dx + e^x dy = -\int_{C_1} (ye^x-1)dx + e^x dy \quad ; \quad \text{on } C_1$$

$x = 0, dx = 0$

$y : 10 \to -10$

$$= -\int_{10}^{-10} dy = -[-10-10]$$

$$= +20$$

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(8) problem 6.9.6 and 6.9.7 : for a simple curve C in the plane, show by Greens theorem that the area enclosed is $A = \frac{1}{2}\oint_C x\,dy - y\,dx$.

Need to prove that $\oint_C x\,dy - y\,dx = 2A$; A: area

let $P = -y$, $Q = x$ $\Rightarrow \frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$

$$\Rightarrow \oint_C x\,dy - y\,dx = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \underbrace{dxdy} = \iint (1+1)\,dxdy = 2\iint dxdy$$

$$= 2A$$

Now we can use this eqn to find the area of ellipse $x = a\cos\theta$, $y = b\sin\theta$; $dx = -a\sin\theta\,d\theta$

$dy = b\cos\theta\,d\theta$

$$A = \frac{1}{2}\oint_{\text{ellipse}} x\,dy - y\,dx = \frac{1}{2}\oint a\cos\theta(b\cos\theta\,d\theta) - b\sin\theta(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\oint ab\cos^2\theta\,d\theta + ab\sin^2\theta\,d\theta$$

$$= \frac{1}{2}\oint ab\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}d\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}ab(2\pi) = \pi ab$$

⑨ **problem 6.10.4:** $\vec{V} = x\cos^2 y\,\hat{i} + xz\,\hat{j} + z\sin^2 y\,\hat{k}$.

evaluate $\oint \vec{V}\cdot d\vec{a}$ over the surface of a sphere with center at origin and radius 3. i.e. $x^2+y^2+z^2=9$

using divergence theorem

$$\oint_S \vec{V}\cdot d\vec{a} = \int_V \nabla\cdot\vec{V}\, dV \quad ; \quad \vec{\nabla}\cdot\vec{V} = \cos^2 y + \sin^2 y = 1$$

$$= \int_V (1)\, dV = \underbrace{\int dV}_{V} = V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (3)^3$$

$$= 36\pi$$

Note that it can be calculated using spherical coordinates

$$\int_V dV = \int_V r^2 dr \sin\theta\, d\theta\, d\phi = \int_0^3 r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{r^3}{3}\Big|_0^3 \times 2 \times 2\pi = \frac{3^3}{3} \times 4\pi = 36\pi$$

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⑩ **problem 6.10.6:** $\vec{V} = (x^3 - x^2)y\,\hat{i} + (y^3 - 2y^2 + y)x\,\hat{i} + (z^2-1)\,\hat{k}$

evaluate $\int_V \vec{\nabla}\cdot\vec{V}\, dV$ unit cube in the first octant

$$\nabla\cdot\vec{V} = y(3x^2-2x) + (3y^2-4y+1) + 2z$$

$$\int_V \nabla\cdot\vec{V}\, dV = \int_V [y(3x^2-2x) + (3y^2-4y+1) + 2z]\, dx\,dy\,dz$$

$$= \int_V y(3x^2-2x)\, dx\,dy\,dz + \int_V (3y^2-4y+1)\, dx\,dy\,dz + \int_V 2z\, dx\,dy\,dz$$

$$= \int_0^1 (3x^2-2x)dx \underbrace{\int_0^1 dy}_{1} \underbrace{\int_0^1 dz}_{1} + \underbrace{\int_0^1 dx}_{1} \int_0^1 (3y^2-4y+1)\,dy \underbrace{\int_0^1 dz}_{1} + 2\underbrace{\int_0^1 dx}_{1}\underbrace{\int_0^1 dy}_{1}\int_0^1 z\,dz$$

$$= \int_0^1 (3x^2-2x)\,dx + \int_0^1 (3y^2-4y+1)\,dy + 2\int_0^1 z\,dz$$

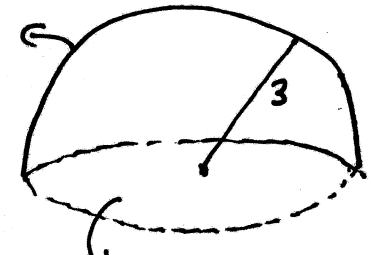
$$= [x^3 - x^2]_0^1 + [y^3 - 2y^2 + y]_0^1 + z^2\Big|_0^1 = 0 + [1-2+1] + 1 = 1$$

(11) problem 6.10.10 : $\vec{V} = y\hat{i} + xz\hat{j} + (2z-1)\hat{k}$

evaluate $\int_S \vec{V}\cdot d\vec{a}$ over the curved surface of the hemisphere

$x^2+y^2+z^2 = 9$ and $z \geq 0$.

curved surface



flat surface

- using divergence theorem

$$\oint_S \vec{V}\cdot d\vec{a} = \int \nabla\cdot\vec{V}\, dV \, ;$$

$\nabla\cdot\vec{V} = 0+0+2$
$= 2$

$$= 2\int dV = 2V$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{4}{3}\pi r^3 \, ; \, r=3$$

$$= \frac{4}{3}\pi 3^3 = 36\pi$$

but $\oint_S \vec{V}\cdot d\vec{a} = \int_{curved} \vec{V}\cdot d\vec{a} + \int_{flat} \vec{V}\cdot d\vec{a} = 36\pi$

$\Rightarrow \int_{curved} \vec{V}\cdot d\vec{a} = 36\pi - \underbrace{\int_{flat} \vec{V}\cdot d\vec{a}}$; but over the flat surface $d\vec{a} = -da\,\hat{k}$ and $z=0$

$\Rightarrow \int \vec{V}\cdot(-\hat{k}\,da) = -\int(2\underset{z=0}{z}-1)\,da$

$= \int da = \pi r^2 = 9\pi$

$= 36\pi - 9\pi$

$= 27\pi$

(12) problem 6.10.11 if $\vec{B} = \nabla\times\vec{A}$, show that

$\oint_S \vec{B}\cdot d\vec{a} = 0$ over any closed surface

$\Rightarrow \oint_S \vec{B}\cdot d\vec{a} = \int_V \nabla\cdot\vec{B}\, dV = \int_V \nabla\cdot(\nabla\times\vec{A})\, dV = \text{zero}$

for any vector field $\vec{A}$. in electromagnetic theory, $\vec{A}$ and $\vec{B}$ are

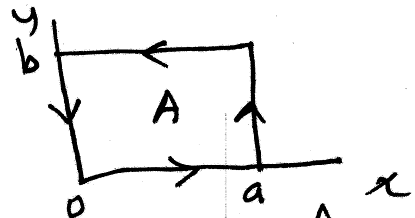
$\vec{A}$: magnetic vector potential

$\vec{B}$: magnetic field

(13) problem 6.11.2 : Given $\vec{A} = (x^2-y^2)\hat{i} + 2xy\,\hat{j}$

a) Find $\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = 4y\,\hat{k}$

b) evaluate $\int \nabla\times\vec{A}\cdot d\vec{a}$ over a rectangle in the x-y plane



$$\int \nabla\times\vec{A}\cdot d\vec{a} = \int (4y\,\hat{k})\cdot(\hat{k}\,da$$
$$= 4\int y\,da = 4\int y\,dx\,dy$$
$$= 4\int_0^a dx \int_0^b y\,dy = 2ab^2$$

$d\vec{a} = da\,\hat{k}$

c) evaluate $\oint_C \vec{A}\cdot d\vec{r}$ around the boundary

$$\oint \vec{A}\cdot d\vec{r} = \oint [(x^2-y^2)\hat{i} + 2xy\hat{j}]\cdot[dx\hat{i}+dy\hat{j}]$$
$$= \oint_C (x^2-y^2)dx + 2xy\,dy$$ moving CCW around the retangle, we got

$$= \underset{y=0,\,dy=0}{\int_0^a} (x^2-y^2)dx + 2xy\,dy + \underset{x=a,\,dx=0}{\int_0^b} (x^2-y^2)dx + 2xy\,dy$$
$$+ \underset{y=b,\,dy=0}{\int_a^0} (x^2-y^2)dx + 2xy\,dy + \underset{x=0,\,dx=0}{\int_b^0} (x^2-y^2)\,dx + 2xy\,dy$$
$$= \int_0^a x^2 dx + 2a\int_0^b y\,dy + \int_a^0 (x^2-b^2)\,dx + 0$$
$$= \frac{a^3}{3} + ab^2 + \left(\frac{x^3}{3} - b^2x\right)\Big|_a^0 = \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2$$
$$= 2ab^2$$

which verifies stocks' theorem

$$\oint \vec{A}\cdot d\vec{r} = \int \nabla\times\vec{A}\cdot d\vec{a}$$

(14) problem 6.11.8: evaluate $\int \nabla \times (x^2 y \hat{i} - xz\hat{k}) \cdot d\vec{a}$ over the closed surface of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

let $\vec{V} = x^2 y \hat{i} - xz\hat{k} \Rightarrow$

$$\oint_S \nabla \times \vec{V} \cdot d\vec{a} = \int_V \nabla \cdot (\nabla \times \vec{V})\, d\tau$$; here we use divergence theorem as the surface is closed

$$= \text{Zero}$$

because $\nabla \cdot (\nabla \times \vec{V})$ is zero for any continuous twice-differentiable vector field $\vec{V}$. we can easily prove this

as
$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 y & 0 & -xz \end{vmatrix} = z\hat{j} - x^2\hat{k}$$

Now $\vec{\nabla} \cdot (\nabla \times \vec{V}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(z\hat{j} - x^2\hat{k}\right) = \frac{\partial z}{\partial y} - \frac{\partial x^2}{\partial z}$
$= 0 - 0 = 0$

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(15) problem 6.11.9: evaluate $\oint_S \vec{V} \cdot d\vec{a}$; with

$\vec{V} = (x + x^2 - y^2)\hat{i} + (2xyz - 2xy)\hat{j} - xz^2\hat{k}$ over the entire surface of the volume in the first octant bounded by $x^2 + y^2 + z^2 = 16$ and the coordinate planes

- since the surface is closed we use divergence theorem

$$\oint_S \vec{V} \cdot d\vec{a} = \int_V \nabla \cdot \vec{V}\, d\tau$$; $\nabla \cdot \vec{V} = (1 + 2x) + (2xz - 2x) - 2xz = 1$

$$= \underbrace{\int_V d\tau}_{V} = V ;$$

$$= \frac{1}{8}\,\frac{4}{3}\pi r^3 = \frac{1}{8}\,\frac{4}{3}\pi (4)^2 = \frac{32}{3}\pi$$

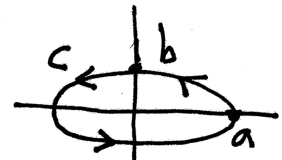
or

$\int^4 \int^{\pi/2} \int^{\pi/2} \ldots \quad 32 \ldots$

(16) problem 6.11.10: let $\vec{v} = 2xy\,\hat{i} + (x^2 - 2x)\,\hat{j} - x^2 z^2 \hat{k}$, find $\int \nabla\times\vec{v}\cdot d\vec{a}$ over the part of surface $z = 9 - x^2 - 9y^2$

- This is an open surface (upside down paraboloid). the projection of this surface on x-y plane gives ellipse (with $z=0$)

$$x^2 + 9y^2 = 9 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{1^2} = 1$$



$a = 3,\ b = 1$

here we use divergence theorem as the surface is open, so

$$\int \nabla\times\vec{v}\cdot d\vec{a} = \oint_c \vec{v}\cdot d\vec{r} = \oint_c 2xy\,dx + (x^2 - 2x)\,dy \text{, with}$$

$x = 3\cos\theta$, $dx = -3\sin\theta\,d\theta$ and $y = \sin\theta$, $dy = \cos\theta\,d\theta$

$0 \leq \theta \leq 2\pi$

$$\Rightarrow \int \nabla\times\vec{v}\cdot d\vec{a} = \oint_c \vec{v}\cdot d\vec{r} = \oint_c 2xy\,dx + (x^2 - 2x)\,dy$$

$$= \oint_c (6\cos\theta\sin\theta)(-3\sin\theta\,d\theta) + (9\cos^2\theta - 6\cos\theta)\cos\theta\,d\theta$$

$$= \int_0^{2\pi} -18\cos\theta\sin^2\theta\,d\theta + 9\int_0^{2\pi}\cos^3\theta\,d\theta - \underbrace{6\int_0^{2\pi}\cos^2\theta\,d\theta}_{\pi}$$

$$= -18\left.\frac{\sin^3\theta}{3}\right|_0^{2\pi} + 0 - 6\pi = -6\pi$$

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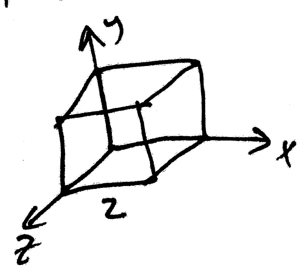
(17) problem 6.11.11: evaluate $\oint \vec{v}\cdot d\vec{a}$; $\vec{v} = (x^2 - y^2)\hat{i} + 3y\hat{j} - 2xz\hat{k}$ over the surface of a cube in the first octant with side length 2.



- the surface is closed ⇒ use stocks' theorem

$$\oint \vec{v}\cdot d\vec{a} = \int_V \nabla\cdot\vec{v}\,d\tau \,; \quad \nabla\cdot\vec{v} = \cancel{2x} + 3 - \cancel{2x} = 3$$

$$= 3\int d\tau = 3V$$

$$= 3(2)^3 = 24$$