Mathematical physics (1)
HW#6 - solution
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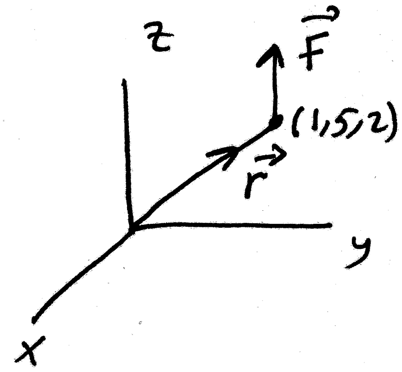
① problem 6.3.7: A force $\vec{F} = 2\hat{i} - 3\hat{j} + \hat{k}$ acts at the point $(1,5,2)$.



a) Find the torque about the origin

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ 1 & 5 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 11\hat{i} + 3\hat{j} - 13\hat{k} = (11, 3, -13)$$

b) Find the torque about the y-axis $(\hat{n} = \hat{j})$

$$\vec{\tau}_y = \hat{n} \cdot \vec{\tau} = \hat{n} \cdot \vec{r} \times \vec{F} = \hat{j} \cdot (11\hat{i} + 3\hat{j} - 13\hat{k}) = 3$$

c) Find the torque about the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$

the line is $(x,y,z) = \underbrace{(0,0,0)}_{\text{point on line}} + \underbrace{(2,1,-2)}_{\vec{A} \text{ parallel vector}} t$

$$\hat{n} = \hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{(2,1,-2)}{\sqrt{4+1+4}} = \frac{1}{3}(2,1,-2)$$

$$\Rightarrow \vec{\tau}_{line} = \hat{n} \cdot \vec{r} \times \vec{F} = \frac{1}{3}(2,1,-2) \cdot (11,3,-13) = \frac{51}{3} = 17$$

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② problem 6.3.12

a) simplify $(\vec{A} \cdot \vec{B})^2 - [(\vec{A} \times \vec{B}) \times \vec{B}] \cdot \vec{A}$

$$(\vec{A} \cdot \vec{B})^2 - \underbrace{[(\vec{A} \cdot \vec{B})\vec{B} - (\vec{A} \cdot \vec{B})\vec{B}] \cdot \vec{A}}_{\text{zero}} = (\vec{A} \cdot \vec{B})^2$$

b) prove Lagrange's identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

starting from L.H.S and using $\vec{F} = \vec{C} \times \vec{D}$, we obtain

$$(\vec{A} \times \vec{B}) \cdot \vec{F} = \vec{A} \cdot (\vec{B} \times \vec{F}) = \vec{A} \cdot (\vec{B} \times \vec{C} \times \vec{D})$$
$$= \vec{A} \cdot [(\vec{B} \cdot \vec{D}) \vec{C} - (\vec{B} \cdot \vec{C}) \vec{D}]$$
$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \checkmark$$

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③ problem 6.3.14 : prove the Jacobi identity

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$$

$$(A \cdot C) B - (A \cdot B) C + (B \cdot A) C - (B \cdot C) A + (C \cdot B) A - (C \cdot A) B = 0 \checkmark$$

where $A \cdot C = C \cdot A$ and $A \cdot B = B \cdot A$, and $C \cdot B = B \cdot C$

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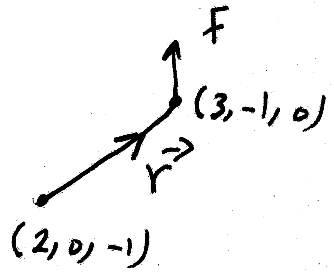
④ problem 6.3.20 (a) the force $\vec{F} = 2\hat{i} - 5\hat{k} = (2, 0, -5)$ acts at the point $(3, -1, 0)$. Find $\vec{\tau}$ about the line

$$(x, y, z) = \underbrace{(2, 0, -1)}_{\text{point on line}} + \underbrace{(0, 3, -4)}_{\vec{A} \text{ parallel vector}} t$$



$$\vec{r} = (3, -1, 0) - (2, 0, -1) = (1, -1, 1)$$

$\vec{\tau}$ about the point $(2, 0, -1)$ is $= \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & 0 & -5 \end{vmatrix} = (5, 7, 2)$
$\vec{r} \times \vec{F}$

now $\hat{n} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{(0, 3, -4)}{\sqrt{25}} = \dfrac{1}{5}(0, 3, -4)$

so the torque about the line passing through $(2, 0, -1)$

is $\hat{n} \cdot (\vec{r} \times \vec{F}) = \dfrac{1}{5}(0, 3, -4) \cdot (5, 7, 2)$

$$= \frac{1}{5}[21 - 8]$$

$$= \frac{13}{5}$$

⑤ problem 6.4.2 : the position vector of a moving particle is given by $\vec{r}(t) = t^2\hat{i} - 2t\hat{j} + (t^2+2t)\hat{k} = (t^2, -2t, t^2+2t)$

a) show that the particle goes through the point $(4,-4,8)$ and at what time does it do this

$(4,-4,8) = (t^2, -2t, t^2+2t)$; remember that $t > 0$

$\Rightarrow 4 = t^2$, $\quad -4 = -2t$, $\quad 8 = t^2 + 2t \Rightarrow t^2 + 2t - 8 = 0$

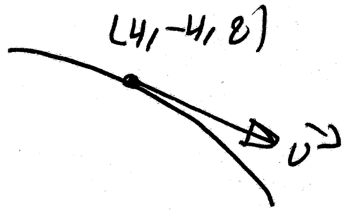
$\Rightarrow t = 2s \quad \Rightarrow t = 2s \quad \Rightarrow t = 2s \quad (t-2)(t+4) = 0$

since all times are equal, then the particle goes through the point $(4,-4,8)$ at a time $t = 2s$

b) find the velocity vector and the speed of the particle at the point $(4,-4,8)$

$\vec{v} = \frac{d\vec{r}}{dt} = (2t, -2, 2t+2)\Big|_{t=2} = (4,-2,6)$; speed $= |\vec{v}| = \sqrt{4^2+(-2)^2+6^2}$
$= \sqrt{56} = 2\sqrt{14}$

c) Find the equation of the line tangent to the curve at the point $(4,-4,8)$



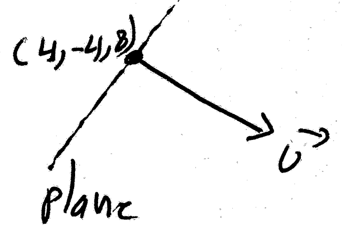
$\vec{r} = \vec{r_0} + \vec{A}t$; $\vec{A}$ (parallel vector) $= \vec{v}$

$(x,y,z) = (4,-4,8) + (4,-2,6)t$

$\Rightarrow x = 4+4t, \quad y = -4-2t, \quad z = 8+6t$

now find the equation of the plane normal to the line above at the point $(4,-4,8)$

$\Rightarrow \vec{N}\cdot(\vec{r} - \vec{r_0}) = 0$; again $\vec{N}$ can be taken as $\vec{v}$



$(4,-2,6)\cdot(x-4, y+4, z-8) = 0$

$\Rightarrow 4x - 2y + 6z = 72$, or divide by 2

$$\boxed{2x - y + 3z = 36}$$

⑥ problem 6.4.4: let $\vec{r}$ be a vector with $|\vec{r}|=1$, prove that either $\vec{r}$ is a constant vector or $\frac{d\vec{r}}{dt}$ is perpendicular to $\vec{r}$

now $\quad \vec{r}\cdot\vec{r} = r^2 = 1 \quad \Rightarrow \frac{d}{dt}(\vec{r}\cdot\vec{r}) = 0$

$\Rightarrow \vec{r}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{r} = 0 \quad \Rightarrow 2\vec{r}\cdot\frac{d\vec{r}}{dt} = 0 \quad \Rightarrow \boxed{\vec{r}\cdot\frac{d\vec{r}}{dt} = 0}$

This means either $\vec{r}$ is a constant vector $\frac{d\vec{r}}{dt} = 0$, or $\frac{d\vec{r}}{dt}$ is $\perp$ to $\vec{r}$ such that

$$\vec{r}\cdot\frac{d\vec{r}}{dt} = |\vec{r}|\,\left|\frac{d\vec{r}}{dt}\right|\cos\theta$$

$\theta = 90°$

$= \text{zero}$

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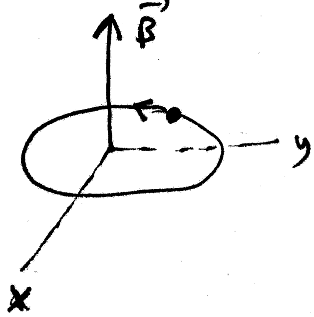
⑦ problem 6.4.6

consider a magnetic field pointing in the $+z$ direction

a particle of mass m and charge q is moving in circle in the x-y plane.

show that the magnetic force $\vec{F} = q\,\vec{v}\times\vec{B}$ and the velocity are perpendicular



$\vec{F} = q\,\vec{v}\times\vec{B}$, but $\vec{F} = m\frac{d\vec{v}}{dt}$

$m\frac{d\vec{v}}{dt} = q\,\vec{v}\times\vec{B}$; dot both sides by $\vec{v}$

$\vec{v}\cdot\left(m\frac{d\vec{v}}{dt}\right) = q\,\vec{v}\cdot\vec{v}\times\vec{B} = \text{zero} \Rightarrow \vec{v}\cdot\underbrace{\left(m\frac{d\vec{v}}{dt}\right)}_{\vec{F}} = 0$

$\Rightarrow \vec{v}\cdot\vec{F} = 0 \quad \Rightarrow \vec{F}$ is $\perp$ to $\vec{v}$

now $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}(v^2)$

$2\vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{d}{dt}(v^2)$

but $\vec{v}$ and $\frac{d\vec{v}}{dt}$ are perpendicular $\Rightarrow \vec{v}\cdot\frac{d\vec{v}}{dt} = 0$

$\Rightarrow 0 = \frac{d}{dt}(v^2) \quad \Rightarrow v^2 = \text{constant}$

$\Rightarrow \quad v = \text{constant}$

now since both v and B are constants, $\Rightarrow \vec{F} = q\,\vec{v}\times\vec{B}$
$= \text{constant}$

8) problem 6.4.9: show that $\frac{d\vec{L}}{dt} = m\vec{r} \times \frac{d^2\vec{r}}{dt^2}$

where $\vec{L}$ is the angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v}) = m\,\vec{r} \times \vec{v}$$

$$\frac{d\vec{L}}{dt} = m\left[\vec{r} \times \frac{d\vec{v}}{dt} + \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} \times \vec{v}\right] = m\left[\vec{r} \times \frac{d\vec{v}}{dt} + \underbrace{\vec{v} \times \vec{v}}_{\text{zero}}\right]$$

$$= m\vec{r} \times \frac{d\vec{v}}{dt} = m\vec{r} \times \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = m\vec{r} \times \frac{d^2\vec{r}}{dt^2} \checkmark$$

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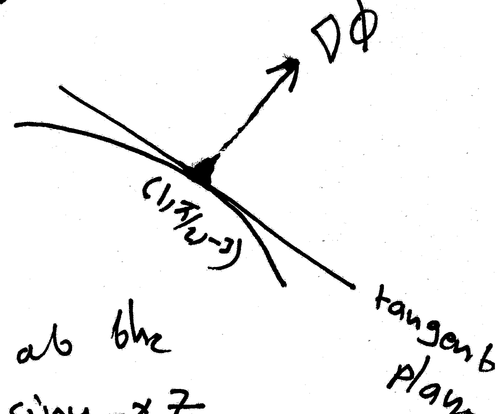
9) problem 6.6.8:

a) Find the directional derivative of $\phi = x^2 + \sin y - xz$ in the direction $\hat{i} + 2\hat{j} - 2\hat{k} = \vec{u}$ at point $(1, \pi/2, -3)$

$\vec{\nabla}\phi = (2x - z)\hat{i} + \cos y\,\hat{j} - x\hat{k} \Rightarrow \vec{\nabla}\phi\big|_{(1,\pi/2,-3)} = (5, 0, -1)$ This direction is the direction of the fastest change in ϕ

$$D_{\hat{u}} = \hat{u} \cdot \nabla\phi\big|_{(1,\pi/2,-3)} = \frac{1}{3}(1, 2, -2)\cdot(5, 0, -1) = \frac{1}{3}[5+2] = 7/3$$



b) $\vec{N}\cdot(\vec{r} - \vec{r_0}) = 0$; $\vec{N} = \vec{\nabla}\phi$

$(5, 0, -1)\cdot(x-1, y-\pi/2, z+3) = 0$

$5(x-1) - (z+3) = 0 \Rightarrow \boxed{5x - z = 8}$

This is the equation of the tangent plane at the point $(1, \pi/2, -3)$ for the case $\phi = 5 = x^2 + \sin y - xz$

- find the equation of the normal line to $\phi = 5$ at the point $(1, \pi/2, -3)$ $\vec{A} = \vec{\nabla}\phi = (5, 0, -1)$

$\vec{r} = \vec{r_0} + \vec{A}t$;

$(x, y, z) = (1, \pi/2, -3) + (5, 0, -1)t$

$$\left.\begin{aligned} \Rightarrow x &= 1 + 5t \\ y &= \pi/2 \\ z &= -3 - t \end{aligned}\right\} \Rightarrow \frac{x-1}{5} = \frac{z+3}{-1} \; ; \; y = \pi/2$$

(10) problem 6.6.13 $\phi(x,y) = e^x \cos y$, where ϕ could be Temperature or electric potential.

a) Find the direction in which temperature is increasing most rapidly at $(1, -\pi/4)$ and the magnitude of the rate of increase

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)e^x \cos y = e^x \cos y\,\hat{i} - e^x \sin y\,\hat{j}$$

$$\vec{\nabla}\phi\Big|_{(1,-\pi/4)} = \frac{e}{\sqrt{2}}\hat{i} + \frac{e}{\sqrt{2}}\hat{j} = \frac{e}{\sqrt{2}}(\hat{i}+\hat{j}) = \frac{e}{\sqrt{2}}(1,1)$$ this is the direction of fastest increase

$$\Rightarrow |\vec{\nabla}\phi| = \frac{e}{\sqrt{2}}\sqrt{1+1} = e$$

b) Find the rate of change of temperature at $(0, \pi/3)$ in the direction of $\hat{i} + \sqrt{3}\,\hat{j}$

$$\Rightarrow \vec{u} = \hat{i} + \sqrt{3}\,\hat{j} = (1, \sqrt{3}) \Rightarrow \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{2}(1, \sqrt{3})$$

Now $$\vec{\nabla}\phi\Big|_{(0,\pi/3)} = \frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j} = \frac{1}{2}(\hat{i} - \sqrt{3}\,\hat{j}) = \frac{1}{2}(1, -\sqrt{3})$$

$$\Rightarrow \hat{u}\cdot\nabla\phi\Big|_{(0,\pi/3)} = \frac{1}{2}(1,\sqrt{3})\cdot\frac{1}{2}(1,-\sqrt{3}) = \frac{1}{4}[1-3] = -\frac{1}{2}$$

c) Find the direction and magnitude of $\vec{E}$ at $(0, \pi)$

$$\vec{E} = -\nabla\phi = -e^x \cos y\,\hat{i} + e^x \sin y\,\hat{j}$$

$$\vec{E}(0,\pi) = -\cos\pi\,\hat{i} + \sin\pi\,\hat{j} = \hat{i} \Rightarrow |\vec{E}(0,\pi)| = 1$$

d) Find $|\vec{E}|$ at $x = -1$ and any y

$$|\vec{E}| = \sqrt{(-e^x \cos y)^2 + (e^x \sin y)^2} = \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}$$

$$= \sqrt{e^{2x}(\cos^2 y + \sin^2 y)} = \sqrt{e^{2x}} = e^x$$

$$|\vec{E}|_{x=-1} = e^{-1} = \frac{1}{e}$$

(11) Find the following gradients

a) $\vec{\nabla} x = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) x = \hat{i}\frac{\partial x}{\partial x} = \hat{i}$

$\vec{\nabla} y = \hat{j}$ and $\vec{\nabla} z = \hat{k}$

b) $\vec{\nabla} r = \vec{\nabla}(x^2+y^2+z^2)^{1/2} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)^{1/2}$

$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)\hat{i} + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)\hat{j} + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)\hat{k}$

$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$

$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\vec{r}}{r}$

c) $\vec{\nabla} r^2 = \nabla(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$= 2(x\hat{i} + y\hat{j} + z\hat{k}) = 2\vec{r}$

d) $\vec{\nabla}\left(\frac{1}{r}\right) = \nabla r^{-1} = \nabla(x^2+y^2+z^2)^{-1/2}$

$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\hat{i} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)\hat{j}$

$- \frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\hat{k}$

$= -\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k} =$

$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}$

e) $\nabla\left(\frac{1}{r^2}\right) = -\frac{2\vec{r}}{r^4}$

in general $\boxed{\nabla r^n = n r^{n-2}\vec{r}}$

(12) problem 6.7.6: find the divergence and curl of the vector $\vec{V} = x^2y\,\hat{i} + y^2x\,\hat{j} + xyz\,\hat{k}$

$$\vec{\nabla}\cdot\vec{V} = 2xy + 2xy + xy = 5xy$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2x & xyz \end{vmatrix} = xz\,\hat{i} - yz\,\hat{j} + (y^2 - x^2)\,\hat{k}$$

(13) problem 6.7.12 Find Laplacian $\nabla^2 (x+y)^{-1}$

$$\nabla^2 (x+y)^{-1} = \frac{\partial^2}{\partial x^2}(x+y)^{-1} + \frac{\partial^2}{\partial y^2}(x+y)^{-1}$$

$$= \frac{2}{(x+y)^3} + \frac{2}{(x+y)^3} = \frac{4}{(x+y)^3}$$

(14) problem 6.7.10 : Find $\nabla^2 \cancel{\ln} (x^2+y^2)$

$$\nabla^2 \ln(x^2+y^2) = \frac{\partial^2}{\partial x^2}\ln(x^2+y^2) + \frac{\partial^2}{\partial y^2}\ln(x^2+y^2)$$

$$= \frac{\partial}{\partial x}\left\{\frac{2x}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right]$$

$$= 2\left\{\frac{\cancel{y^2 - x^2}}{(x^2+y^2)^2} + \frac{\cancel{x^2 - y^2}}{(x^2+y^2)^2}\right\} = \text{zero}$$

(15) problem 6.7.16: Find $\nabla^2 \ln(x^2+y^2+z^2)$

$$\nabla^2 \ln(x^2+y^2+z^2) = \nabla^2 \ln r^2$$

; now using $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$ we obtain with $f(r) = \ln r^2$

$$= \frac{-2}{r^2} + \frac{2}{r}\,\frac{2}{r} = \frac{2}{r^2}$$

$$= \frac{2}{(x^2+y^2+z^2)}$$

(16) problem 6.7.18: if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Find $\nabla \times (\vec{k} \times \vec{r})$; where $\vec{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$

now $\vec{k} \times \vec{r} = \begin{vmatrix} i & j & k \\ k_x & k_y & k_z \\ x & y & z \end{vmatrix} = i(zk_y - yk_z) - \hat{j}(zk_x - xk_z) + \hat{k}(yk_x - xk_y)$

so $\nabla \times (\vec{k} \times \vec{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (zk_y - yk_z) & (-zk_x + xk_z) & (yk_x - xk_y) \end{vmatrix}$

$= i(k_x + k_x) - \hat{j}(-k_y - k_y) + \hat{k}(k_z + k_z)$

$= 2k_x \hat{i} + 2k_y \hat{j} + 2k_z \hat{k} = 2\vec{k}$

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(17) problem 6.7.19 Find $\vec{\nabla} \cdot \frac{\vec{r}}{r}$; $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

using $\nabla \cdot (\phi \vec{v}) = \vec{v} \cdot \nabla \phi + \phi (\nabla \cdot \vec{v})$, we obtain with $\phi = \frac{1}{r}$ and $\vec{v} = \vec{r}$

$\nabla \cdot \frac{\vec{r}}{r} = \vec{r} \cdot \nabla \frac{1}{r} + \frac{1}{r} \nabla \cdot \vec{r} = \vec{r} \cdot \left(\frac{-\vec{r}}{r^3}\right) + \frac{3}{r}$

$= -\frac{\vec{r} \cdot \vec{r}}{r^3} + \frac{3}{r} = \frac{-r^2}{r^3} + \frac{3}{r} = -\frac{1}{3} + \frac{3}{r} = \frac{2}{r}$

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(18) problem 6.7.20! Find $\nabla \times \frac{\vec{r}}{r}$; $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

using $\vec{\nabla} r = \frac{\vec{r}}{r}$, we get

$\nabla \times \frac{\vec{r}}{r} = \nabla \times (\nabla r) = \text{zero}$ as $\nabla \times \nabla \phi = 0$ for any scalar function ϕ

or using identity $\nabla \times (\phi \vec{v}) = \phi \nabla \times \vec{v} + (\nabla \phi) \times \vec{v}$

$\nabla \times \frac{1}{r}\vec{r} = \frac{1}{r} \underbrace{\nabla \times \vec{r}}_{0} + \underbrace{\nabla\left(\frac{1}{r}\right)}_{\frac{-\vec{r}}{r^3}} \times \vec{r} = \frac{-\vec{r}}{r^3} \times \vec{r} = \text{zero}$