Mathematical physics (1)
HW #5 - Solution
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① <u>problem 3.8.2</u>: determine wether the following vectors are dependent or independent. if they are dependent find the linearly independent subset.

$\vec{A} = (1,-2,3)$; $\vec{B} = (1,1,1)$; $\vec{C} = (-2,1,-4)$; $\vec{D} = (3,0,5)$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ -2 & 1 & -4 \\ 3 & 0 & 5 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

indicates linear dependence

$\Rightarrow \vec{A}, \vec{B}, \vec{C}, \vec{D}$ are linearly dependent

the basis vectors are $u = \hat{i} - 2\hat{j} + 3\hat{k} = (1,-2,3)$

and $v = 3\hat{j} - 2\hat{k} = (0,3,-2)$

now $\vec{y} = f\vec{u} + g\vec{v} = f(1,-2,3) + g(0,3,-2) = (f, -2f+3g, 3f-2g)$

i) let $\vec{y} = \vec{A} = (1,-2,3) = (f, -2f+3g, 3f-2g) \Rightarrow f=1$ and

$-2f+3g = -2 \Rightarrow -2+3g = -2 \Rightarrow g=0$

$\Rightarrow \vec{A} = \vec{u} = (1,-2,3)$ ✓

ii) let $\vec{y} = \vec{B} = (1,1,1) = (f, -2f+3g, 3f-2g) \Rightarrow f=1$ and $g=1$

$\Rightarrow \vec{B} = \vec{u} + \vec{v} = (1,-2,3) + (0,3,-2) = (1,1,1)$ ✓

iii) let $\vec{y} = \vec{C} = (-2,1,-4) = (f, -2f+3g, 3f-2g) \Rightarrow f=-2,\ g=-1$

$\Rightarrow \vec{C} = -2\vec{u} - \vec{v} = -2(1,-2,3) - (0,3,-2) = (-2,1,-4)$ ✓

iiii) let $\vec{y} = \vec{D} = (3,0,5) = (f, -2f+3g, 3f-2g) \Rightarrow f=3,\ g=2$

$\Rightarrow \vec{D} = 3\vec{u} + 2\vec{v} = 3(1,-2,3) + 2(0,3,-2) = (3,0,5)$ ✓

② problem 3.8.10 check linear dependence of the functions e^{ix}, e^{-ix} using wronskin method

$$\begin{vmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{vmatrix} = -2i \Rightarrow \text{they are linearly independent}$$

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③ problem 3.8.15 $e^x, e^{ix}, \cosh x \Rightarrow$

$$\begin{vmatrix} e^x & e^{ix} & \cosh x \\ e^x & ie^{ix} & \sinh x \\ e^x & -e^{ix} & \cosh x \end{vmatrix} = 2e^x e^{ix}[\sinh x - \cosh x]$$

$$= 2e^x e^{ix}\left[\frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2}\right]$$

$$= -2e^{ix} \quad \text{linearly independent}$$

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④ problem 3.9.1 : show that $(AB)C = ABC$

using index notations

$$[(AB)C]_{ij} = \sum_k (AB)_{ik} C_{kj} \quad ; \text{ but } (AB)_{ik} = \sum_l A_{il} B_{lk}$$

$$= \sum_k \sum_l A_{il} B_{lk} C_{kj} = (ABC)_{ij}$$

$$\therefore (AB)C = (ABC)$$

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⑤ problem 3.9.5: show that AA^T is symmetric matrix

$(AA^T)^T = (A^T)^T A^T = AA^T$; when I used $(AB)^T = B^T A^T$

$\therefore (A \cdot A^T)^T = AA^T \Rightarrow AA^T$ is symmetric

since for symmetric matrix, the matrix is equal to its transpose

6 problem 3.9.17:

(a) Show that if A and B are symmetric, then AB is not symmetric unless A and B commute.

If A and B are symmetric, then $A = A^T$ and $B = B^T$. We now examine

$$(AB)^T = B^T A^T = BA\,,$$

after using the fact that A and B are symmetric matrices. We conclude that $(AB)^T = AB$ if and only if $AB = BA$. $\boxed{\text{That is, } AB \text{ is not symmetric unless } A \text{ and } B \text{ commute.}}$

(b) Show that a product of orthogonal matrices is orthogonal.

Consider orthogonal matrices Q_1 and Q_2. By definition [cf. the table at the top of p. 138 of Boas], we have $Q_1^{-1} = Q_1^T$ and $Q_2^{-1} = Q_2^T$. We now compute

$$(Q_1Q_2)^{-1} = Q_2^{-1}Q_1^{-1} = Q_2^T Q_1^T = (Q_1Q_2)^T\,, \tag{4}$$

after using the fact that Q_1 and Q_2 are orthogonal. In deriving Eq. (4), we have used the following properties of the inverse and the transpose

$$(AB)^{-1} = B^{-1}A^{-1}\,, \quad \text{and} \quad (AB)^T = B^T A^T\,,$$

for any pair of matrices A and B. Thus, we have shown that $\boxed{(Q_1Q_2)^{-1} = (Q_1Q_2)^T, \text{ which implies that } Q_1Q_2 \text{ is orthogonal.}}$

(c) Show that if A and B are Hermitian, then AB is not Hermitian unless A and B commute.

If A and B are Hermitian, then $A = A^\dagger$ and $B = B^\dagger$. We now examine

$$(AB)^\dagger = B^\dagger A^\dagger = BA\,, \tag{5}$$

after using the fact that A and B are Hermitian matrices. In deriving Eq. (5), we have used the fact that:

$$(AB)^\dagger = ((AB)^*)^T = (A^* B^*)^T = (B^*)^T (A^*)^T = B^\dagger A^\dagger\,. \tag{6}$$

We conclude that $\boxed{(AB)^\dagger = AB \text{ if and only if } AB = BA.}$ That is, AB is not Hermitian unless A and B commute.

(d) Show that a product of unitary matrices is unitary.

Consider unitary matrices U_1 and U_2. By definition [cf. the table at the top of p. 138 of Boas], we have $U_1^{-1} = U_1^\dagger$ and $U_2^{-1} = U_2^\dagger$. We now compute

$$(U_1U_2)^{-1} = U_2^{-1}U_1^{-1} = U_2^\dagger U_1^\dagger = (U_1U_2)^\dagger\,,$$

after using the fact that U_1 and U_2 are unitary and employing the property of the Hermitian conjugation given in Eq. (6). Thus, we have shown that $\boxed{(U_1U_2)^{-1} = (U_1U_2)^\dagger,}$ which implies that U_1U_2 is orthogonal.

⑦ problem 3.9.19 (a) prove that $Tr(AB) = Tr(BA)$

$Tr(AB) = \sum_i (AB)_{ii}$; but $(AB)_{ij} = \sum_k A_{ik} B_{kj}$

let $j \to i$

$(AB)_{ii} = \sum_k A_{ik} B_{ki}$

$$= \sum_i \sum_k A_{ik} B_{ki}$$

$$= \sum_k \sum_i B_{ki} A_{ik} .$$

$$= \sum_k (BA)_{kk} = Tr(BA)$$

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⑧ problem 3.9.23: show that the following matrices are hermitian wether A is Hermitian or not

i) $AA^\dagger$

$$\Rightarrow (AA^\dagger)^\dagger = (A^\dagger)^\dagger A^\dagger = AA^\dagger \checkmark$$

where I used $(AB)^\dagger = B^\dagger A^\dagger$

ii) $A + A^\dagger$

$$\Rightarrow (A + A^\dagger)^\dagger = A^\dagger + (A^\dagger)^\dagger = A^\dagger + A = A + A^\dagger \checkmark$$

iii) $i(A - A^\dagger)$

$$[i(A - A^\dagger)]^\dagger = (A - A^\dagger)^\dagger (i)^\dagger = (A^\dagger - A^{\dagger\dagger})(-i)$$

$$= (A^\dagger - A)(-i) = i(A - A^\dagger) \checkmark$$

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⑨ problem 3.11.9: show that $\det(C^{-1}MC) = \det(M)$?

Now $\det(CC^{-1}) = \det(C^{-1}) \cdot \det(C)$

$$\det(I) = \det(C^{-1}) \cdot \det(C)$$

$$1 = \det(C^{-1}) \cdot \det(C)$$

$\det(C^{-1}) = \dfrac{1}{\det(C)}$

$\Rightarrow$ follow

$$\det(C^{-1}MC) = \det(C^{-1}) \cdot \det(M) \cdot \det(C) = \frac{\det(M) \cdot \cancel{\det(C)}}{\cancel{\det(C)}}$$

$$= \det(M) \checkmark$$

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(10) <u>problem 3.11.10</u> : show that

$$\mathrm{Tr}(C^{-1}MC) = \mathrm{Tr}(M)$$

$$\Rightarrow \mathrm{Tr}(C^{-1}MC) = \mathrm{Tr}(CC^{-1}M) = \mathrm{Tr}(IM) = \mathrm{Tr}(M) \checkmark$$

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(11) <u>problem 3.11.12:</u> Find the eigenvalues and the normalized eigenvectors of the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

$$\Rightarrow \det(A - \lambda I) = 0 \quad \Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0 \quad \Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + \lambda^2 = 0 \quad \Rightarrow \quad (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4, -1$$

i) for $\lambda = 4 \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $\begin{matrix} x + 3y = 4x & \text{--}(1) \\ 2x + 2y = 4y & \text{--}(2) \end{matrix}$

from (1) and (2) we get $x = y$, set $x = 1 \Rightarrow y = 1$

eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, normalized eigenvector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ii) for $\lambda = -1 \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $\begin{matrix} x + 3y = x & \text{--}(3) \\ 2x + 2y = -y & \text{--}(4) \end{matrix}$

from (3) and (4), we get $y = -\frac{2}{3}x$

set $x = 3 \Rightarrow y = -2$

eigenvector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$; normalized eigenvector $\frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

note that eigenvectors are not orthogonal as the original matrix (A) is Not orthogonal matrix ($A^{-1} \neq A^T$)

12 **problem 3.11.16!** Find the eigenvalues and normalized eigenvectors of $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda)(-1-\lambda) + 2[0 - 2(2-\lambda)] = 0$$

$$\Rightarrow 4(2-\lambda) + (2-\lambda)(2-\lambda)(1+\lambda) = 0 \Rightarrow (2-\lambda)[4 + (2-\lambda)(1+\lambda)] = 0$$

either $2-\lambda = 0 \Rightarrow \boxed{\lambda = +2}$ or

$$4 + (2-\lambda)(1+\lambda) = 0 \Rightarrow \lambda^2 - \lambda - 6 = 0 \Rightarrow (\lambda-3)(\lambda+2) = 0$$

$$\Rightarrow \lambda = 3, -2$$

$\therefore \boxed{\lambda = 2, 3, -2}$ Three eigenvalues

i) for $\lambda = 2$ $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$

$2x + 2z = 2x$ --(1)

$2y = 2y$ --(2)

$2x - z = 2z$ --(3)

from (1), $z = 0$. from (3), $x = 0$. from (2) y is arbitrary

set $y = 1 \Rightarrow$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ normalized eigenvectors

ii) for $\lambda = 3 \Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$

$2x + 2z = 3x$ --(4)

$2y = 3y$ --(5)

$2x - z = 3z$ --(6)

from (5), $y = 0$. from (4) and (6) $z = \frac{x}{2}$, set $x = 2 \Rightarrow z = 1$

$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ normalized $\Rightarrow \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

iii) for $\lambda = -2 \Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$

$2x + 2z = -2x$ --(7)

$2y = -2y$ --(8)

$2x - z = -2z$ --(9)

from (8), $y = 0$.

from (7) and (9), $z = -2x$. set $x = 1 \Rightarrow z = -2$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \rightarrow$ normalize $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. Note that all eigenvectors are orthogonal, Although A is not orthogonal matrix ($A^{-1} \neq A^T$)

(13) <u>problem 3.11.43:</u> verify that the following matrix is hermitian. Find its eigenvalues and eigenvectors. write a unitary matrix which diagonlizes the matrix

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 2 \end{pmatrix} \; ; \; A^{+} = (A^{*})^{T} = \begin{pmatrix} 1 & 2i \\ -2i & -2 \end{pmatrix} \Rightarrow A \text{ is hermitian as } A = A^{+}$$

Now
$$\begin{vmatrix} 1-\lambda & 2i \\ -2i & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda) - 4 = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$(\lambda-2)(\lambda+3) = 0 \Rightarrow \boxed{\lambda = 2, -3} \text{ eigenvalues}$$

i) for $\lambda = 2$

$$\begin{pmatrix} 1 & 2i \\ -2i & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{aligned} x + 2iy &= 2x \quad \text{--(1)} \\ -2ix - 2y &= 2y \quad \text{--(2)} \end{aligned}$$

from (1) and (2), $2iy = x \Rightarrow y = -\frac{i}{2}x$. Set $x = 2 \Rightarrow y = -i$

$\Rightarrow \begin{pmatrix} 2 \\ -i \end{pmatrix} \rightarrow$ normalize $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -i \end{pmatrix}$ first eigenvector

ii) for $\lambda = 2$

$$\begin{pmatrix} 1 & 2i \\ -2i & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left.\begin{aligned} x + 2iy &= -3x \quad \text{--(3)} \\ -2ix - 2y &= -3y \quad \text{--(4)} \end{aligned}\right\} \Rightarrow y = 2ix$$

set $x = 1$ $\Rightarrow y = 2i$

$\begin{pmatrix} 1 \\ 2i \end{pmatrix} \rightarrow$ normalize $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ second eigenvector

$$\Rightarrow U = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -i & 2i \end{pmatrix} \rightarrow \text{Find } U^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & i \\ 1 & -2i \end{pmatrix}$$

Now
$$A_{diag} = U^{-1}AU = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & i \\ 1 & -2i \end{pmatrix}\begin{pmatrix} 1 & 2i \\ -2i & -2 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -i & 2i \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 4 & 2i \\ -3 & 6i \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -i & 2i \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

(14) problem 3.11.24! consider $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Find eigenvalues and eigenvectors.

and Find unitary matrix that diagonlizes A

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \boxed{\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0}$$ cubic eq^n

solve by google cubic root calculator $\Rightarrow \lambda = 8, \underbrace{-1, -1}_{\text{repeated eigenvalues}}$

i) for $\lambda = 8$

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{aligned} 3x + 2y + 4z &= 8x \quad \text{---(1)} \\ 2x + 2z &= 8y \quad \text{--(2)} \\ 4x + 2y + 3z &= 8z \quad \text{--(3)} \end{aligned}$$

subtract (3)−(1) $\Rightarrow x - z = 8z - 8x \Rightarrow z = x$. from (2)

$2x + 2x = 8y \Rightarrow y = \frac{x}{2}$. set $x = 2 \Rightarrow z = 2$ and $y = 1$

$$\Rightarrow \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \rightarrow \text{normalize } \frac{1}{\sqrt{9}}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \vec{r_1}$$

ii) $\lambda = -1$

$$\Rightarrow \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{aligned} 3x + 2y + 4z &= -x \\ 2x + 2z &= -y \\ 4x + 2y + 3z &= -z \end{aligned}$$

or $\left.\begin{aligned} 4x + 2y + 4z &= 0 \\ 2x + y + 2z &= 0 \\ 4x + 2y + 4z &= 0 \end{aligned}\right\} \Rightarrow$ basically one eq^n so x, y, z can't be determined. so set $x = 0$

$\Rightarrow 2y + 4z = 0 \Rightarrow z = -\frac{y}{2}$. set $y = 2$

$\Rightarrow z = -1$

$$\Rightarrow \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \rightarrow \text{normalize } \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \vec{r_2}$$

iii) $\lambda = -1$, second repeated eigenvalue. The third eigenvector must be orthogonal to both $\vec{r_1}$ and $\vec{r_2}$ i.e

$$\vec{r_3} = \vec{r_1} \times \vec{r_2} = \frac{1}{3\sqrt{5}}\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 0 & 2 & -1 \end{vmatrix} = \frac{1}{3\sqrt{5}}\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$$ already normalized

define $U = \begin{pmatrix} \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{-1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{pmatrix}$ and check that $A_{diag} = U^{-1}AU = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$