Mathematical physics (1)
HW #4 - Solution
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① problem 3.6.7

Find $\underset{1\times 2}{(2 \quad 3)} \underset{2\times 2}{\begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} = \underset{1\times 2}{(4 \quad 5)} \underset{2\times 1}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} = -4+10 = 6$

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② problem 3.6.9 $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$; $B = \begin{pmatrix} 10 & 4 \\ -5 & -2 \end{pmatrix}$

Find AB and BA

$AB = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{null matrix}}$ and $BA = \begin{pmatrix} 22 & 44 \\ -11 & -22 \end{pmatrix} \Rightarrow AB \neq BA$

see that $AB = 0$, but neither A nor B is zero. Note also that $\det(AB) = 0$, so AB is singular, meaning it does not have inverse.

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③ problem 3.6.10 consider

$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$; $C = \begin{pmatrix} 7 & 6 \\ 2 & 3 \end{pmatrix}$; $D = \begin{pmatrix} -3 & 2 \\ 7 & 5 \end{pmatrix}$

show that $AC = AD$

$\Rightarrow AC = AD = \begin{pmatrix} 11 & 12 \\ 33 & 36 \end{pmatrix}$

see that $AC = AD$, but $C \neq D$ and $A \neq 0$

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④ problem 3.6.21 solve the following equations by the method of finding the inverse of the coefficient matrix

$$\begin{aligned} x+2z &= 8 \\ 2x-y &= -5 \\ x+y+z &= 4 \end{aligned} \Rightarrow \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{r} = \underbrace{\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix}}_{k}$$

$\therefore$ $Mr = k \Rightarrow r = M^{-1}k = \dfrac{C^T}{|M|} k$; where

$$|M| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 5 \text{ and } C = \begin{pmatrix} -1 & -2 & 3 \\ 2 & -1 & -1 \\ 2 & 4 & -1 \end{pmatrix}; \; C^T = \begin{pmatrix} -1 & 2 & 2 \\ -2 & -1 & 4 \\ 3 & -1 & -1 \end{pmatrix}$$

$$\Rightarrow r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 2 & 2 \\ -2 & -1 & 4 \\ 3 & -1 & -1 \end{pmatrix}\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -10 \\ 5 \\ 25 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

$\Rightarrow x = -2, \; y = 1, \; z = 5$

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⑤ problem 3.6.30 If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 1st Pauli matrix, find $\sin(kA)$

using $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, we have

$$\boxed{\text{Taylor Series } f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=0} (x)^n}$$

$\sin(kA) = kA - \dfrac{k^3A^3}{3!} + \dfrac{k^5A^5}{5!} - \dfrac{k^7A^7}{7!} + \cdots$, but

$A^2 = AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A^3 = A^2A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$,

and $A^5 = A^3A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$, $A^7 = A^5A^2 = A$, so on

(odd power), so

$\therefore$ $A^3 = A^5 = A^7 = A^9 = \cdots = A$

$\sin(kA) = kA - \dfrac{k^3A}{3!} + \dfrac{k^5A}{5!} - \dfrac{k^7A}{7!} + \cdots = A \sin k$ ✓

⑥ check the linearity of the following functions, vector functions, and operators.

<u>problem 3.7.1</u>: $f(\vec{r}) = \vec{A}\cdot\vec{r} + 3$

$$f(\vec{r_1}+\vec{r_2}) = \vec{A}\cdot(\vec{r_1}+\vec{r_2}) + 3 = \vec{A}\cdot\vec{r_1} + \vec{A}\cdot\vec{r_2} + 3 \neq f(\vec{r_1}) + f(\vec{r_2})$$

$\Rightarrow f(\vec{r})$ is Not linear ✓

<u>problem 3.7.2</u>: $f(\vec{r}) = \vec{A}\cdot(\vec{r} - z\hat{k})$

$$f(\vec{r_1}+\vec{r_2}) = \vec{A}\cdot(\vec{r_1}+\vec{r_2} - z_1\hat{k} - z_2\hat{k})$$
$$= \vec{A}\cdot(\vec{r_1} - z_1\hat{k}) + \vec{A}\cdot(\vec{r_2} - z_2\hat{k}) = f(\vec{r_1}) + f(\vec{r_2})$$

and $f(a\vec{r}) = \vec{A}\cdot(a\vec{r} - az\hat{k}) = a\,\vec{A}\cdot(\vec{r} - z\hat{k})$
$= a\,f(\vec{r})$

$\Rightarrow f(\vec{r})$ is Linear ✓

<u>problem 3.7.5</u>: $\vec{F}(\vec{r}) = \vec{A}\times\vec{r}$

$$\vec{F}(\vec{r_1}+\vec{r_2}) = \vec{A}\times(\vec{r_1}+\vec{r_2}) = \vec{A}\times\vec{r_1} + A\times\vec{r_2} = \vec{F}(\vec{r_1}) + \vec{F}(r_2)$$

and $\vec{F}(a\vec{r}) = \vec{A}\times(a\vec{r}) = a\,\vec{A}\times\vec{r} = a\,\vec{F}(\vec{r}) \Rightarrow$

$\Rightarrow \vec{F}(\vec{r})$ is Linear ✓

<u>problem 3.7.7</u>: $\int_0^1 (\text{functions})\,dx$

$\int_0^1 [g(x) + f(x)]\,dx = \int_0^1 g(x)\,dx + \int_0^1 f(x)\,dx$ ✓ , and

$\int_0^1 a\,g(x)\,dx = a\int_0^1 g(x)\,dx$ ✓ $\Rightarrow$ Linear

<u>problem 3.7.8</u>: $f(x) = \ln x$; operate on positive numbers

$\ln(a+b) \neq \ln a + \ln b \Rightarrow$ Not Linear

Problem 3.7.9: $f(x) = x^2$

$$f(x_1+x_2) = (x_1+x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 \neq f(x_1) + f(x_2)$$

$\Rightarrow$ Not Linear

Problem 3.7.10:

$$f(x) = \frac{1}{x} \Rightarrow f(x_1+x_2) = \frac{1}{x_1+x_2} \neq \frac{1}{x_1} + \frac{1}{x_2}$$

$\Rightarrow f(x)$ is Not Linear

Problem 3.7.12: $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, ... and so on

all these operators are Linear

$\frac{d}{dx}$: $\frac{d}{dx}[f(x)+g(x)] = \frac{df}{dx} + \frac{dg}{dx}$ and $\frac{d}{dx}(a f(x)) = a\frac{df}{dx}$

$\Rightarrow \frac{d}{dx}$ is Linear

similarly

$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d^2f}{dx^2} + \frac{d^2g}{dx^2}$ and $\frac{d^2}{dx^2} a f(x) = a\frac{d^2f}{dx^2}$

$\Rightarrow \frac{d^2}{dx^2}$ is Linear ...

Problem 3.7.13 (b): is $x^2D^2 - 2xD + 7$ linear operator

where $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$

$$\Rightarrow [x^2D^2 - 2xD + 7][f(x)+g(x)] = x^2(f''+g'') - 2x(f'+g') + 7(f+g)$$
$$= x^2f'' - 2xf' + 7f + x^2g'' - 2xg' + 7g$$
$$= [x^2D^2 - 2xD + 7]f + [x^2D^2 - 2xD + 7]g$$

and

$$[x^2D^2 - 2xD + 7][a f(x)] = a[x^2f'' - 2xf' + 7f] = a[x^2D^2 - 2xD + 7]f$$

⑦ problem 3.7.23 : consider the active transformation of a vector in 2D. The transformation is represented by the matrix $A = \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$. show that the transformation is orthogonal, find the determinant and angle of rotation

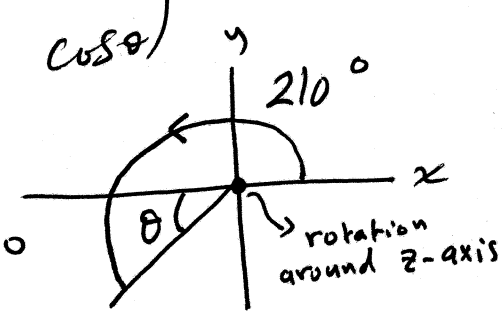
Now $AA^T = \frac{1}{2} \times \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, so it is orthogonal

$|A| = \begin{vmatrix} -\frac{\sqrt{3}}{2} & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{vmatrix} = \frac{3}{4} + \frac{1}{4} = 1$ ⇒ so this is rotation

to find angle of rotation compare A with R_θ of active rotation

$A = R_\theta \Rightarrow \begin{pmatrix} -\frac{\sqrt{3}}{2} & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = -1/2$

$\Rightarrow \theta = 30^\circ$ or $\theta = 180 + 30 = 210^\circ$

⑧ problem 3.7.26 : let $A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ be an active transformation in 2D (x,y) plane. show that A is orthogonal, find the determinant and line of reflection

Now $|A| = \left(\frac{1}{5}\right)^2(-9-16) = \frac{-25}{25} = -1$, so it is reflection now to find the axis of reflection, let A operate on a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ directed along the axis of reflection, so that the vector will not be affected by this transformation, so $Ar = r$

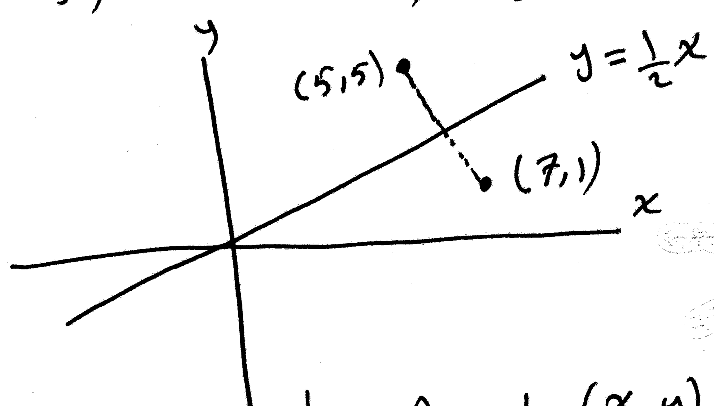
$$\begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} \frac{3}{5}x + \frac{4}{5}y = x \quad ---(1) \\ \frac{4}{5}x - \frac{3}{5}y = y \quad ---(2) \end{array}$$

multiply (1) by 3 and (2) by 4 and then add, we get

$$5x = 3x + 4y \Rightarrow 2x = 4y \Rightarrow x = 2y \text{ or } y = \tfrac{1}{2}x$$

- to understand the reflection consider the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ now under reflection through the line $y = \frac{1}{2}x$, this points transforms to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} x' = 7 \\ y' = -1 \end{array}$$



See that the point $(x, y) = (2, 1)$ is located on the line of reflection, so it should not be affected by this transformation ; check!!

$$\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ as expected

⑧ problem 3.7.29 construct the matrix corresponding to rotation of $90°$ around y-axis followed by reflection through the x-z plane

$$\therefore R_{x-z} R_y(\theta=90) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \checkmark$$

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⑨ problem 3.7.32 Find the axis and angle of rotation of the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

- see that $\det(A) = +1$, so this is a rotation. Now to find the axis of rotation, set $Ar = r$ as any vector on the axis of rotation is not affected by rotation, so

$$Ar = r \Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} -z = x \\ -y = y \Rightarrow 2y = 0 \quad \boxed{y=0} \\ -x = z \end{matrix}$$

either x or z is arbitrary, so set $x = 1 \Rightarrow z = -1$

$\Rightarrow$ the axis of rotation is $(1, 0, -1) = \hat{i} - \hat{k}$, use to find the angle of rotation

$\text{trace}(A) = 2\cos\theta + 1 \Rightarrow \text{trace}(A) = -1 = 2\cos\theta + 1$

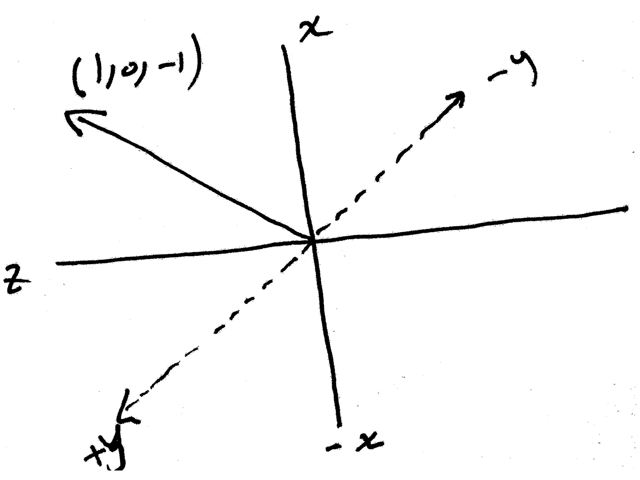
$\Rightarrow 2\cos\theta = -2 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi$

check:

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$x \rightarrow -z$

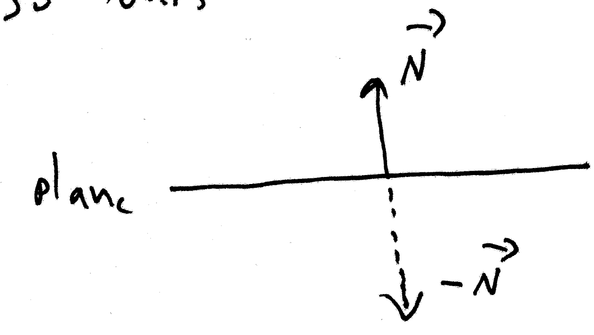
or $\hat{i} \rightarrow -\hat{k}$



(10) problem 3.7.34 let $A=\begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$ be a matrix that represent a reflection about some plane. Find the plane of reflection

First see that $\det(A) = -1$, so this is a reflection.

the normal vector $\vec{N} = (x,y,z)$ is reversed by this matrix, so



$A N = -N$

$$\begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-x\\-y\\-z\end{pmatrix} \Rightarrow \begin{array}{l} x=-x \\ -z=-y \\ -y=-z \end{array} \Rightarrow \begin{array}{l} 2x=0 \Rightarrow x=0 \\ \text{let } y=1 \\ \Rightarrow z=1 \end{array}$$

$\Rightarrow \vec{N} = (0,1,1)$

now to find the equation of the plane of reflection use $\vec{N}\cdot(\vec{r}-\vec{r_0}) = 0$; where $\vec{r_0}$ is any point on the plane that can be taken for simplicity $(0,0,0)$ $\therefore$ $\vec{r_0} = (0,0,0)$

$\Rightarrow \vec{N}\cdot\vec{r} = 0$

$\Rightarrow (0,1,1)\cdot(x,y,z) = 0 \Rightarrow$

$\boxed{y+z=0}$ yz plane