# Mathematical physics (1)
# HW #3 – solution
# Dr. Gassem Alzoubi

① problem 3.2.3 solve $x-2y+13=0$ and $y-4x=17$

rewrite $\Rightarrow$ $\begin{aligned} x-2y&=-13 \\ -4x+y&=17 \end{aligned}$ $\Rightarrow A=\left(\begin{array}{cc|c} 1 & -2 & -13 \\ -4 & 1 & 17 \end{array}\right)$

let $R_2 \to R_2 + 4R_1 \Rightarrow A=\left(\begin{array}{cc|c} 1 & -2 & -13 \\ 0 & -7 & -35 \end{array}\right)$ ---- (1)

so $-7y=-35 \Rightarrow y=5$ and $x-2y=-13 \Rightarrow x-2\times5=-13$

$x=-13+10$

$=-3$

$\therefore$ $\boxed{x=-3 \text{ and } y=5}$

Note that we can further row reduce eq (1) as follows

$\frac{R_2}{-7} \Rightarrow A=\begin{pmatrix} 1 & -2 & -13 \\ 0 & 1 & 5 \end{pmatrix}$ ; Now let us make

$A_{12}=-2$ zero by $R_1 \to R_1+2R_2 \Rightarrow A=\left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array}\right)$

which gives $x=-3$ and $y=5$ $\Rightarrow$ unique solution

as $\text{Rank}(M)=\text{Rank}(A)=2=n$

② problem 3.2.5 solve using row reduction method

$\left.\begin{aligned} 2x+y-z&=2 \\ 4x+2y-2z&=3 \end{aligned}\right\} \Rightarrow A=\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 2 & -2 & 3 \end{array}\right)$

let $R_2 \to R_2 - 2R_1 \Rightarrow A=\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right)$

see that $\text{Rank}(M)=1 < \text{Rank}(A)=2 \Rightarrow$ inconsistent

$\Rightarrow$ No solution

③ problem 3.2.6 solve using row reduction method

$$\left.\begin{array}{l} x+y-z=1 \\ 3x+2y-2z=3 \end{array}\right\}$$ two equations with three unknowns $(n=3)$

$$\Rightarrow A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 2 & -2 & 3 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_1} A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_2} A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x=1 \text{ and} \\ -y+z=0 \Rightarrow y=z \end{array}$$

see that $\mathrm{Rank}(M) = 2 = \mathrm{Rank}(A) < n=3$

④ problem 3.2.12 solve

$$\left.\begin{array}{l} 2x+5y+z=2 \\ x+y+2z=1 \\ x+0y+5z=3 \end{array}\right\} A = \left(\begin{array}{ccc|c} 2 & 5 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 5 & 3 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & 5 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 2 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_1 + R_3} \left(\begin{array}{cccc} 0 & 3 & -3 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right)$$ ; divide $R_1$ by 3 and $R_3$ by 2

$$\Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 0 & 0 & -1 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$\Rightarrow y=1$, $-z=-1 \Rightarrow z=1$, and

$x+y+2z=1 \Rightarrow x+1+2=1 \Rightarrow x=-2$

$\therefore \boxed{x=-2,\ y=1,\ z=1}$

Note that $\mathrm{Rank}(M) = \mathrm{Rank}(A) = 3 = n$

⑤ problem 3.3.) Find $A = \begin{vmatrix} -2 & 3 & 4 \\ 3 & 4 & -2 \\ 5 & 6 & -3 \end{vmatrix}$, expand using $1^{st}$ row

$$\det(A) = (-2)(-1)^{1+1}\begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} + (3)(-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix}$$

$$+ 4(-1)^{1+3}\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= (-2)[-12+12] - 3(-9+10) + 4(18-20) = -3-8 = -11$$

the same answer can be obtained using row reduction method

$$\therefore A = \begin{vmatrix} -2 & 3 & 4 \\ 3 & 4 & -2 \\ 5 & 6 & -3 \end{vmatrix} \xrightarrow{C_3 \to C_3 + 2C_1} \begin{vmatrix} -2 & 3 & 0 \\ 3 & 4 & 4 \\ 5 & 6 & 7 \end{vmatrix} \xrightarrow{C_2 \to C_2 + \frac{3}{2}C_1}$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 3 & \frac{17}{2} & 4 \\ 5 & \frac{27}{2} & 7 \end{vmatrix}$$ ; now expand using the $1^{st}$ row (Laplace development)

$$\Rightarrow \det(A) = -2(-1)^{1+1}\begin{vmatrix} \frac{17}{2} & 4 \\ \frac{27}{2} & 7 \end{vmatrix} = (-2)\left[\frac{17}{2}\times 7 - \frac{27}{2}\times 4\right] = -11$$

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⑥ problem 3.3.15 use cramer's rule to solve

$$\left.\begin{aligned} x - 2y &= 4 \\ 5x + z &= 7 \\ x + 2y - z &= 3 \end{aligned}\right\} \Rightarrow D = \begin{vmatrix} 1 & -2 & 0 \\ 5 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -14$$

$$\Rightarrow D_x = \begin{vmatrix} 4 & -2 & 0 \\ 7 & 0 & 1 \\ 3 & 2 & -1 \end{vmatrix} = -28, \quad D_y = \begin{vmatrix} 1 & 4 & 0 \\ 5 & 7 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 14, \quad D_z = \begin{vmatrix} 1 & -2 & 4 \\ 5 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 42$$

$$\Rightarrow x = \frac{D_x}{D} = \frac{-28}{-14} = 2 \;; \quad y = \frac{D_y}{D} = \frac{14}{-14} = -1, \quad z = \frac{D_z}{D} = \frac{42}{-14} = -3$$

⑦ problem 3.3.16 use cramer's rule to show that $|A|^2 = 1$ for the following equations

$$\left.\begin{aligned} A - B &= -1 \\ ikA - \kappa B &= ik \end{aligned}\right\} \quad D = \begin{vmatrix} 1 & -1 \\ ik & -\kappa \end{vmatrix} = -\kappa + ik = -(\kappa - ik)$$

and $D_A = \begin{vmatrix} -1 & -1 \\ ik & -\kappa \end{vmatrix} = \kappa + ik$

$$\Rightarrow A = \frac{D_A}{D} = \frac{\kappa + ik}{-(\kappa - ik)} = -\frac{\kappa + ik}{\kappa - ik}$$

Now $|A|^2 = A\bar{A} = \cancel{\dfrac{\kappa + ik}{\kappa - ik}} \cdot \cancel{\dfrac{\kappa - ik}{\kappa + ik}} = 1$

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⑧ problem 3.4.15: consider the vector $\vec{A} = 2\hat{i} - \hat{j} + 2\hat{k}$

a) Find a unit vector in the same direction as $\vec{A}$

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} = \frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$$

b) find a vector in the same direction as $\vec{A}$ but of magnitude 12.

any parallel vector to $\vec{A}$ must have the same sign of unit vectors. let the vector $\vec{B}$ be // to $\vec{A}$ such that $\vec{B} = B_x\hat{i} - B_y\hat{j} + B_z\hat{k}$ such that $\sqrt{B_x^2 + B_y^2 + B_z^2} = 12$. So we need $B_x^2 + B_y^2 + B_z^2 = 144$.

try $(B_x, B_y, B_z) = (8, 4, 8) \Rightarrow \vec{B} = 8\hat{i} - 4\hat{j} + 8\hat{k}$

check that $\vec{A} \times \vec{B} = 0$ ; $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 8 & -4 & 8 \end{vmatrix}$

$$\Rightarrow \vec{A} \times \vec{B} = \hat{i}(-8+8) - \hat{j}(16-16) + \hat{k}(-8+8) = \text{zero}$$

c) Find a vector perpendicular to $\vec{A}$. let $\vec{C} \perp \vec{A}$

$\Rightarrow \vec{A}\cdot\vec{C} = 0$ , where $\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

$\Rightarrow 2C_x - C_y + 2C_z = 0 \Rightarrow$ equation of a plane. any vector on this plane is $\perp$ to $\vec{A}$. any set of numbers $(C_x, C_y, C_z)$ that satisfy the equation of plane is a good choice. try $\vec{C} = (C_x, C_y, C_z) = (1, 4, 1)$

i.e $\vec{C} = \hat{i} + 4\hat{j} + \hat{k}$. of course there are many other choices

check! $\vec{A}\cdot\vec{C} = (2\hat{i} - \hat{j} + 2\hat{k})\cdot(\hat{i} + 4\hat{j} + \hat{k})$

$= 2 - 4 + 2 = 0$ ✓

d) find a unit vector $\perp$ to $\vec{A}$.

$$\hat{C} = \frac{\vec{C}}{|\vec{C}|} = \frac{\hat{i} + 4\hat{j} + \hat{k}}{\sqrt{1^2 + 4^2 + 1^2}} = \frac{1}{\sqrt{18}}(\hat{i} + 4\hat{j} + \hat{k})$$

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(9) problem 3.4.18 consider $\vec{A} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{B} = 5\hat{i} + 2\hat{j} - 2\hat{k}$

a) show that $\vec{A} \perp B$ (i.e orthogonal)

$\vec{A}\cdot\vec{B} = 10 - 2 - 8 =$ zero ✓ so $\vec{A} \perp \vec{B}$

b) find a third vector perpendicular to both $\vec{A}$ and $\vec{B}$

let the vector $\vec{C}$ be perpendicular to $\vec{A}$ and $\vec{B}$, then

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ 5 & 2 & -2 \end{vmatrix} = -6\hat{i} + 24\hat{j} + 9\hat{k} \checkmark$$

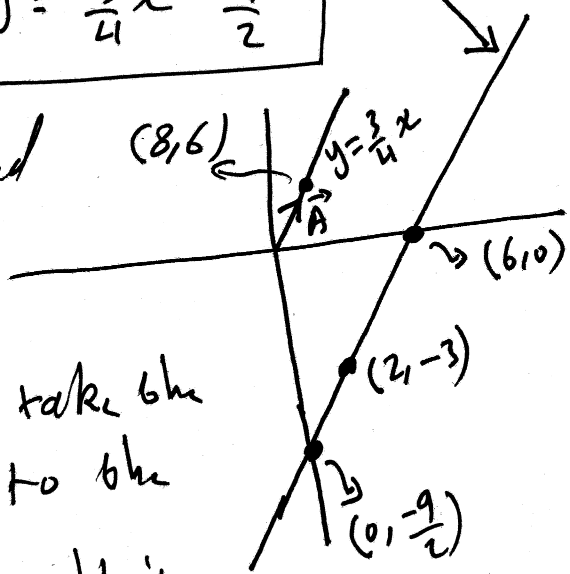
you can divide $\vec{C}$ by $-3$ to get $\vec{C}' = 2\hat{i} - 8\hat{j} - 3\hat{k}$ ✓

(10) problem 3.5.1 : write the vector equation for the line through (2,-3) with slope 3/4.

the equation of the line with slope 3/4 is $y = \frac{3}{4}x + a$

to find the intercept a, plug in the point (2,-3) $\Rightarrow$

$-3 = \frac{3}{4} \times 2 + a \Rightarrow a = -\frac{9}{2} \Rightarrow \boxed{y = \frac{3}{4}x - \frac{9}{2}}$




now to find the vector equation of this line we need another parallel vector.

any vector with the same slope will do the job, so take the line $y = \frac{3}{4}x$ that is parallel to the line $y = \frac{3}{4}x - \frac{9}{2}$. Now to obtain a vector on the parallel line ($y = \frac{3}{4}x$), pick up any point, say $x = 8 \Rightarrow y = 6 \Rightarrow$ the parallel vector $\vec{A}$ is given by $\vec{A} = 8\hat{i} + 6\hat{j} = (8,6)$.

$\Rightarrow \vec{r} = \vec{r_0} + t\vec{A}$

$(x,y) = (x_0,y_0) + t\vec{A}$

$(x,y) = (2,-3) + t(8,6)$ } vector eq$^n$.

Now the symmetric eq$^{ns}$ are given by

$\left.\begin{array}{l} x = 2 + 8t \\ y = -3 + 6t \end{array}\right\} \Rightarrow t = \frac{x-2}{8} = \frac{y+3}{6}$

$\therefore \boxed{\frac{x-2}{8} = \frac{y+3}{6}}$ symmetric equations

(11) problem 3.5.2: find the slope of the line whose parametric equation is $\vec{r} = (\hat{i}-\hat{j}) + (2\hat{i}+3\hat{j})t$

$\Rightarrow (x\hat{i}+y\hat{j}) = (\hat{i}-\hat{j}) + t(2\hat{i}+3\hat{j}) \Rightarrow x = 1+2t$, and
$y = -1+3t$

$\Rightarrow t = \frac{x-1}{2}$, substitute in $y = -1+3t$
$y = -1 + 3\left(\frac{x-1}{2}\right) = \frac{3}{2}x - \frac{5}{2}$

$\Rightarrow$ slope $= \frac{3}{2}$

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(12) problem 3.5.4: write in parametric form, the eqn of a line that is $\perp$ to the line $\vec{r_1} = (2\hat{i}+4\hat{j}) + (\hat{i}-2\hat{j})t$ and passes through $(1,0)$

$\vec{r_1} = (2\hat{i}+4\hat{j}) + t(\hat{i}-2\hat{j}) \Rightarrow x_1 = 2+t$ and $y_1 = 4-2t$
$\Rightarrow t = x_1 - 2 \Rightarrow y_1 = 4-2(x_1-2)$
$\boxed{y_1 = 6-2x_1}$
$m_2 = -2$

now recall that for any two perpendicular lines with slopes $m_1$ and $m_2$, we have
$m_2 = -\frac{1}{m_1}$, so the expected line must have a slope of $\frac{1}{2}$

$\Rightarrow y_2 = \frac{1}{2}x_2 + a$ ; to find $a$, substitute the point $(1,0)$ in $y_2 \Rightarrow 0 = \frac{1}{2} + a \Rightarrow a = -\frac{1}{2}$

$\Rightarrow \boxed{y_2 = \frac{1}{2}x_2 - \frac{1}{2}}$ ; Now to find a parallel vector to this line, pick up any two points; say $(0,-1/2)$ and $(2,1/2)$

$\Rightarrow \vec{A} = (2-0, 1/2-(-1/2)) = (2,1)$ this is not unique; any other points would work, so

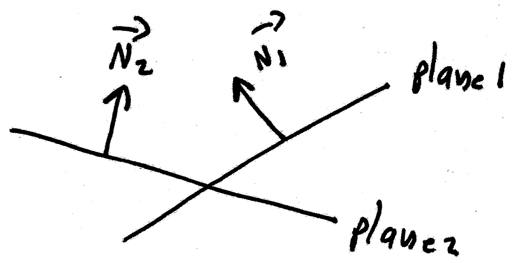
$\vec{r_2} = \vec{r_0} + t\vec{A} \Rightarrow (x,y) = (1,0) + t(2,1)$

or $x = 1+2t$
$y = t$

(13) problem 3.5.28: Find the equation of the plane through $(-4,-1,2)$ and perpendicular to the following planes

$2x - y - z = 4 \Rightarrow N_1 = (2,-1,-1)$

$3x - 2y - 6z = 7 \Rightarrow N_2 = (3,-2,-6)$




the direction of the plane is Normal to both $\vec{N_1}$ and $\vec{N_2}$

$$\Rightarrow \vec{N} = \vec{N_1} \times \vec{N_2} = \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 3 & -2 & -6 \end{vmatrix} = 4\hat{i} + 9\hat{j} - \hat{k} = (4, 9, -1)$$

$$\Rightarrow \vec{N} \cdot (\vec{r} - \vec{r_0}) = 0 \Rightarrow (4,9,-1) \cdot (x+4, y+1, z-2) = 0$$

$$\Rightarrow 4(x+4) + 9(y+1) - 1(z-2) = 0 \Rightarrow \boxed{4x + 9y - z = -27}$$

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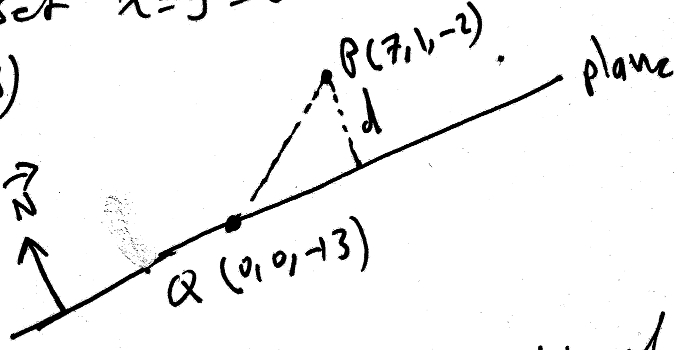
(14) problem 3.5.29: Find a point on the plane $2x - y - z = 13$. Find the distance from $(7,1,-2)$ to the plane.

first to find a point on the plane set $x = y = 0$ and find $z$

$\Rightarrow z = -13 \Rightarrow Q(0,0,-13)$




now the vector $\vec{QP} = (7,1,11)$ and the normal to the plane is $\vec{N} = (2,-1,-1)$

so the distance, d, from P to the plane is projection of $\vec{QP}$ on $\hat{N}$

$d = |\vec{QP} \cdot \hat{N}|$ ; where $\hat{N} = \frac{1}{\sqrt{6}}(2,-1,-1)$

$$\Rightarrow \vec{QP} \cdot \hat{N} = (7,1,11) \cdot \frac{1}{\sqrt{6}}(2,-1,-1) = \frac{1}{\sqrt{6}}\left[14 - 1 - 11\right]$$

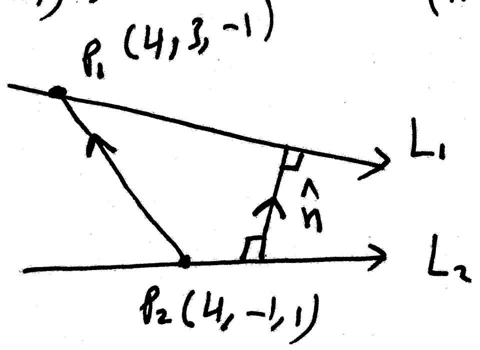
$$\Rightarrow d = |\vec{QP} \cdot \hat{N}|$$

$$d = \frac{1}{\sqrt{6}}|14 - 1 - 11| = \frac{1}{\sqrt{6}}|2| = \frac{2}{\sqrt{6}}$$

(15) problem 3.5.41 : Find the shortest distance between the following two lines

$L_1: \ (x,y,z) = (4,3,-1) + t(1,1,1) \quad ; \ \vec{A_1} = (1,1,1)$ and $L_1 \not\parallel L_2$

$L_2: \ (x,y,z) = (4,-1,1) + t(1,-2,-1) \ ; \ \vec{A_2} = (1,-2,-1)$ (Not parallel)




Note that the two lines are above each other, so they don't intersect. Now need to find $\hat{n}$ which is normal to both $\vec{A_1}$ and $\vec{A_2}$

$$\vec{n} = \vec{A_1} \times \vec{A_2} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \hat{i} + 2\hat{j} - 3\hat{k} = (1,2,-3)$$

$\Rightarrow \hat{n} = \frac{1}{\sqrt{14}}(1,2,-3) \Rightarrow$ the distance $d$ is projection of $\vec{P_1P_2}$ on $\hat{n}$

$d = |\vec{P_1P_2} \cdot \hat{n}| \ ;$ when $\vec{P_1P_2} = (0,4,-2)$

$$= \frac{1}{\sqrt{14}}(0+8+6) = \frac{14}{\sqrt{14}} = \sqrt{14}$$

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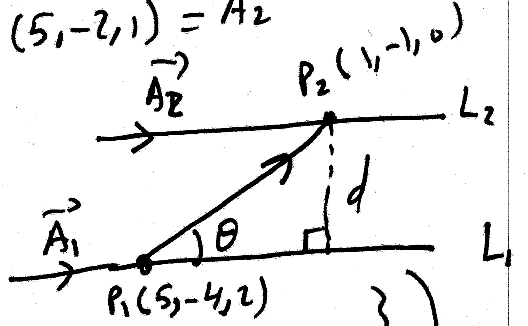
(16) problem 3.5.33: Find the distance between the two parallel lines

$L_1: \ (x,y,z) = (5,-4,2) + t\overbrace{(5,-2,1)}^{\vec{A_1}} \Rightarrow \vec{A_1} = (5,-2,1) = \vec{A_2}$

$L_2: \ (x,y,z) = (1,-1,0) + t\underbrace{(5,-2,1)}_{\vec{A_2}}$




$\Rightarrow \vec{P_1P_2} = (-4,3,-2)$

$\hat{A_1} = \hat{A_2} = \frac{1}{\sqrt{30}}(5,-2,1)$

$$d = |\vec{P_1P_2} \times \hat{A_1}| = \frac{1}{\sqrt{30}} \begin{vmatrix} i & j & k \\ -4 & 3 & -2 \\ 5 & -2 & 1 \end{vmatrix} = \frac{1}{\sqrt{30}} |(-\hat{i} - 6\hat{j} - 7\hat{k})|$$

$$= \frac{1}{\sqrt{30}} \sqrt{(-1)^2 + (-6)^2 + (-7)^2} = \sqrt{\frac{86}{30}} = \sqrt{\frac{43}{15}}$$

$\sin\theta = \frac{d}{|\vec{P_1P_2}|}$

$\Rightarrow d = |\vec{P_1P_2}| \sin\theta$
$= |\vec{P_1P_2} \times \hat{A_1}|$