Mathematical Physics (1)

HW #2 – Solution

Dr. Gassem Alzoubi

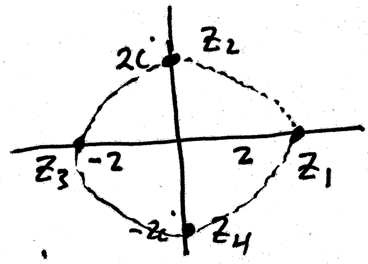
---

① <u>problem 2.10.4:</u> Find the roots of $\sqrt[4]{16}$

$\sqrt[4]{16}$: $z = 16 \Rightarrow x = 16, y = 0 \Rightarrow r = 16, \ \theta = \tan^{-1}\left(\frac{0}{16}\right) = 0$

$$\Rightarrow \sqrt[4]{16} = (16)^{1/4}\left[\cos\left(\frac{0+2\pi k}{4}\right) + i\sin\left(\frac{0+2\pi k}{4}\right)\right] \Rightarrow \theta = 0 \quad \Rightarrow 4\text{-roots} \quad k = 0,1,2,3$$

$k = 0 \Rightarrow z_1 = (2^4)^{1/4}\left[\cos 0) + i\sin(0)\right] = 2$



the other 3 roots set on a circle of radius $r = 2$ and $\frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$ apart

we can deduce them to be; $2i, -2, -2i$, or can be calculated from the above eqn:

$k = 1 \Rightarrow z_2 = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2i$

$k = 2 \Rightarrow z_3 = 2\left[\cos(\pi) + i\sin(\pi)\right] = -2$

$k = 3 \Rightarrow z_4 = 2\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] = -2i$

Note the roots sum to zero $\quad \sum \text{roots} = 0$

i.e. $2 + 2i - 2 - 2i = \text{zero}$

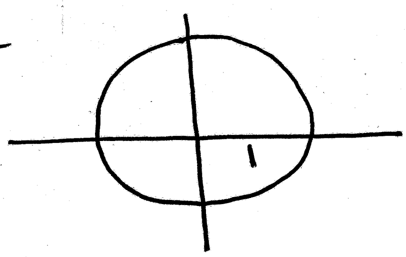
---

② problem 2.10.19 Find $\sqrt[3]{i}$; $z = i \Rightarrow x = 0, y = 1, \theta = \frac{\pi}{2}$

$$\sqrt[3]{i} = re^{i\theta} = (1)e^{i\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)} = \cos\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)$$

we have 3 roots located on a circle of radius $r = 1$ and displaced by $\frac{2\pi}{3}$; $\quad k = 0, 1, 2$

let us find them



$k=0 \Rightarrow z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{1}{2}(\sqrt{3}+i)$

$k=1 \Rightarrow z_2 = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$; but $\frac{5\pi}{6} = \frac{6\pi}{6} - \frac{\pi}{6} = \pi - \frac{\pi}{6}$

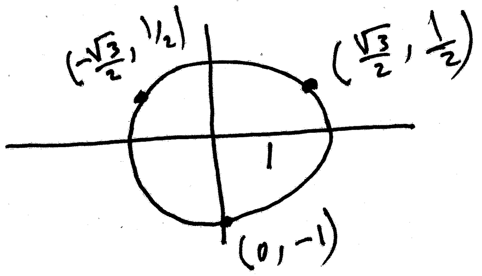
$= -\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{1}{2}(-\sqrt{3}+i)$

$k=2 \Rightarrow z_3 = \cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)$; but $\frac{9\pi}{6} = \frac{6\pi}{6} + \frac{3\pi}{6}$

$= \pi + \frac{\pi}{2}$

$= \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)$

$= -i$



so the 3-roots are

$-i, \ \frac{1}{2}(\sqrt{3}+i), \ 1/2(-\sqrt{3}+i)$

---

③ problem 2.10.28: find the formulas for $\sin 3\theta$ and $\cos 3\theta$

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta \cdot i\sin\theta + 3\cos\theta \cdot i^2\sin^2\theta + i^3\sin^3\theta$

using DeMoivre's formula

where I used $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow \cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta$

equate real parts $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$

$= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ ✓

and equate imaginary parts

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$ ✓

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(4) problem 2.11.14 : show that $\int_0^{2\pi} \sin^2 4x \, dx = \pi$

$$\int_0^{2\pi} \sin^2 4x \, dx = \int_0^{2\pi} \sin 4x \cdot \sin 4x \, dx$$

$$= \int_0^{2\pi} \frac{e^{i4x} - e^{-i4x}}{2i} \cdot \frac{e^{i4x} - e^{-i4x}}{2i} \, dx = -\frac{1}{4}\left[\int_0^{2\pi} (e^{i8x} - 1 - 1 + e^{-i8x}) \, dx\right.$$

$$= -\frac{1}{4}\left[\int_0^{2\pi} -2\,dx + \int_0^{2\pi} e^{i8x} \, dx + \int_0^{2\pi} e^{-i8x} \, dx\right\}$$

$$= -\frac{1}{4}\left[-2x\Big|_0^{2\pi} + \frac{e^{i8x}}{i8}\Big|_0^{2\pi} + \frac{e^{-i8x}}{-i8}\Big|_0^{2\pi}\right\}$$

$$= -\frac{1}{4}\left[-4\pi + \frac{1}{i8}(e^{i16\pi} - 1) - \frac{1}{i8}(e^{-i16\pi} - 1)\right\}$$

$$= -\frac{1}{4}\left[-4\pi + \frac{e^{i16\pi}}{i8} - \cancel{\frac{1}{i8}} - \frac{e^{-i16\pi}}{i8} + \cancel{\frac{1}{i8}}\right\} \quad ; \; 16\pi \equiv 2\pi \equiv 0$$

$$= -\frac{1}{4}\left[-4\pi + \frac{1}{i8}(1-0) - \frac{1}{i8}(1-0)\right\} = -\frac{1}{4}[-4\pi] = \pi$$

---

(5) problem 2.11.10 evaluate $\sin(i \ln i)$

using $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, we get

$$\sin(i \ln i) = \frac{e^{ii\ln i} - e^{-ii\ln i}}{2i} = \frac{e^{-\ln i} - e^{\ln i}}{2i} = -\frac{1}{2i}(e^{\ln i} - e^{-\ln i})$$

$$= -\frac{1}{2i}\cdot(i - (i)^{-1}) = -\frac{1}{2i}\cdot\left(i - \frac{1}{i}\right)$$

$$= -\frac{1}{2i}\cdot(i + i) = -\frac{1}{2i}\cdot(2i) = -1$$

⑥ problem 2.12.8: show that $\cosh 2z = \cosh^2 z + \sinh^2 z$

$$\cosh^2 z + \sinh^2 z = \left(\frac{e^{z}+e^{-z}}{2}\right)^2 + \left(\frac{e^{z}-e^{-z}}{2}\right)^2$$

$$= \frac{1}{4}\left[(e^{2z}+e^{-2z}+2) + (e^{2z}+e^{-2z}-2)\right]$$

$$= \frac{1}{4}\left\{2e^{2z}+2e^{-2z}\right\} = \frac{e^{2z}+e^{-2z}}{2} = \cosh 2z \checkmark$$

---

⑦ problem 2.12.16 show that $\tan iz = i \tanh z$

using $\tan x = -i \dfrac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}$, let $x = iz$

$$\Rightarrow \tan iz = -i \frac{e^{iiz}-e^{-iiz}}{e^{iiz}+e^{-iiz}} = -i \frac{e^{-z}-e^{+z}}{e^{-z}+e^{+z}}$$

$$= i \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}} = i \tanh z \checkmark$$

---

⑧ problem 2.12.20: show that

$e^{nz} = (\cosh z + \sinh z)^n = \cosh nz + \sinh nz$ and

$e^{-nz} = (\cosh z - \sinh z)^n$

- start from $\cosh z = \dfrac{e^{z}+e^{-z}}{2} \Rightarrow e^{z}+e^{-z} = 2\cosh z$ --(1)

and $\sinh z = \dfrac{e^{z}-e^{-z}}{2} \Rightarrow e^{z}-e^{-z} = 2\sinh z$ --(1)

add (1) + (2) $\Rightarrow e^{z} = \cosh z + \sinh z \Rightarrow e^{nz} = (\cosh z + \sinh z)^n$

subtracte (1) − (2) $\Rightarrow e^{-z} = \cosh z - \sinh z \Rightarrow e^{-nz} = (\cosh z - \sinh z)^n$

- now recall that Demov formula

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$

let $\theta \to -iz \Rightarrow$

$$\Rightarrow \left(\underbrace{\cos(-iz)}_{\text{even}} + i\sin(-iz)\right)^n = \underbrace{\cos(n(-iz))}_{\text{even}} + i\sin(n(-iz))$$

$$(\cosh z - i^2 \sinh z)^n = \cosh nz - i^2 \sinh nz$$

$$\boxed{(\cosh z + \sinh z)^n = \cosh nz + \sinh nz} \quad \text{--(3)}$$

similarly, if one lets $\theta \to iz$, then

$$\boxed{(\cosh z - \sinh z)^n = \cosh nz - \sinh nz} \quad \text{--(4)}$$

subbing $n=3$ in equations (3) and (4), we get

$$(\cosh z + \sinh z)^3 = \cosh 3z + \sinh z \quad \text{---(5)}, \text{ and}$$

$$(\cosh z - \sinh z)^3 = \cosh 3z - \sinh z \quad \text{---(6)}$$

adding (5) + (6), we get

$$2\cosh 3z = (\cosh z + \sinh z)^3 - (\cosh z - \sinh z)^3$$

with little algebra, one gets

$$\cosh 3z = \cosh^3 z + 3\cosh z \sinh^2 z$$

similarly subtracting (5) − (6), we get

$$2\sinh 3z = (\cosh z + \sinh z)^3 - (\cosh 3z - \sinh 3z)^3$$

with little algebra, one gets

$$\sinh 3z = 3\cosh^2 z \sinh z + \sinh^3 z$$

---

⑨ problem 2.12.27: find $|\sin(4+3i)|$

$$\sin(4+3i) = \sin 4 \cos 3i + \cos 4 \sin 3i = \sin 4 \cosh 3 + i\cos 4 \sinh 3$$

$$= -7.62 - 6.55i \;;$$ Put your calculator on Radians system

$$\Rightarrow |\sin(4+3i)| = \sqrt{(-7.62)^2 + (-6.55)^2} = 10.05$$

(10) problem 2. 14.1 Find $\ln(-e)$

$z=-e \Rightarrow x=-e,\ y=0,\ r=e,\ \theta=\pi$

$\Rightarrow \ln(-e) = \ln e + i\pi = 1+i\pi$; take only $k=0$

(11) problem 2.14.7 find $\ln\left(\frac{1+i}{1-i}\right)$;

$$\ln\left(\frac{1+i}{1-i}\right) = \ln\left(\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}}\right) = \ln e^{i\pi/2} = i\frac{\pi}{2}\ln e$$

$$= i\frac{\pi}{2}$$

(12) problem 2. 14. 13 find $i^{\frac{2i}{\pi}}$

$$\Rightarrow i^{\frac{2i}{\pi}} = e^{\frac{2i}{\pi}\ln i} = e^{\frac{2i}{\pi}(i(\frac{\pi}{2}+2\pi k))}$$; take $k=0$ only

$$= e^{-1} = \frac{1}{e} \approx (2.72)^{-1} = 0.37$$

(13) problem 2.14.18 find $\cos(2i\ln i)$

$$\cos(2i\ln i) = \frac{e^{i2i\ln i}+e^{-i2i\ln i}}{2}$$; using $\ln i = i\frac{\pi}{2}+2\pi k$ and taking $k=0$ $\Rightarrow \ln i = i\pi/2$

$$= \frac{e^{-2i\frac{\pi}{2}}+e^{+2i\pi/2}}{2}$$

$$= \frac{e^{-i\pi}+e^{i\pi}}{2} = \cosh(i\pi)$$; but $\cosh y = \cos iy$, so

$$= \cos(i\,i\pi) = \cos(-\pi) = \cos\pi$$

$$=-1$$

(14) problem 2.17.19 show that $\tan^{-1} z = \frac{1}{2i} \ln \frac{1+iz}{1-iz}$

let $w = \tan^{-1} z \Rightarrow z = \tan w = -i \dfrac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$

let $u = e^{iw}$

$\Rightarrow z = -i \dfrac{u - u^{-1}}{u + u^{-1}} \Rightarrow uz + u^{-1} z = -iu + iu^{-1}$, multiply by u

$\Rightarrow u^2 z + z = -iu^2 + i \Rightarrow (z+i)u^2 = i - z$

$\Rightarrow u^2 = \dfrac{i - z}{z + i} = \dfrac{i - z}{i + z} = \dfrac{i(1 - \frac{z}{i})}{i(1 + \frac{z}{i})} = \dfrac{1 + iz}{1 - iz}$

but $u^2 = e^{2iw}$

$\Rightarrow e^{2iw} = \dfrac{1+iz}{1-iz} \Rightarrow 2iw = \ln \dfrac{1+iz}{1-iz} \Rightarrow$

$w = \tan^{-1} z = \dfrac{1}{2i} \ln \dfrac{1+iz}{1-iz}$

---

(15) problem 2.17.22 show that $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$

let $w = \tanh^{-1} z \Rightarrow z = \tanh w = \dfrac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$

$\Rightarrow z = \dfrac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$; let $u = e^{w}$

$\Rightarrow z = \dfrac{u - u^{-1}}{u + u^{-1}} \Rightarrow uz + u^{-1} z = u - u^{-1}$, multiply by u

$u^2 z + z = u^2 - 1 \Rightarrow u^2(z-1)$

$u^2(z-1) = -1 - z \Rightarrow u^2 = \dfrac{1+z}{1-z}$

but $u^2 = e^{2w}$

$\Rightarrow e^{2w} = \dfrac{1+z}{1-z} \Rightarrow 2w = \ln \dfrac{1+z}{1-z}$

$\Rightarrow w = \tanh^{-1} z = \dfrac{1}{2} \ln\left(\dfrac{1+z}{1-z}\right)$