Mathematical physics (1)
HW #10 - Solution
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① **problem 8.5.1**: solve $y'' + y' - 2y = 0$

$\Rightarrow r^2 + r - 2 = 0 \Rightarrow (r-1)(r+2) = 0 \Rightarrow r_1 = 1,\ r_2 = -2$

$\Rightarrow y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{x} + c_2 e^{-2x}$

② **problem 8.5.2**: solve $y'' - 4y' + 4y = 0$

$\Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r-2)(r-2) = 0 \Rightarrow r_1 = r_2 = 2$ repeated roots

$\Rightarrow y = (c_1 + c_2 x) e^{2x} = c_1 e^{2x} + c_2 x e^{2x}$

③ **problem 8.5.3**: solve $y'' + 9y = 0$

$\Rightarrow r^2 + 9 = 0 \Rightarrow r^2 = -9 \Rightarrow r = \pm 3i,\quad r_1 = 3i;\ r_2 = -3i$

$\alpha = 0,\ \beta = 3$

$\Rightarrow y(x) = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$

$= c_1 e^{3ix} + c_2 e^{-3ix}$ or $= c_3 \cos 3x + c_4 \sin 3x$

④ **problem 8.5.10**: solve $y'' - 2y' = 0$

$\Rightarrow r^2 - 2r = 0 \Rightarrow r(r-2) = 0 \Rightarrow r_1 = 0,\ r_2 = 2$

$\Rightarrow y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 + c_2 e^{2x}$

⑤ problem 8.5.19: solve $y'' + (1+2i)y' + (i-1)y = 0$

$\Rightarrow r^2 + (1+2i)r + (i-1) = 0 \Rightarrow r = \dfrac{-(1+2i) \pm \sqrt{(1+2i)^2 - 4(i-1)}}{2}$

$\Rightarrow r = \dfrac{-1-2i \pm \sqrt{(1+2i)(1+2i) - 4i + 1}}{2} = \dfrac{-1-2i \pm \sqrt{-3+4i-4i+4}}{2}$

$= \dfrac{-1-2i \pm 1}{2} \Rightarrow r_1 = -i, \; r_2 = -1-i = -(1+i)$

Note that r_2 is not complex conjugate of r_1, so the solution is written as

$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1^{-ix} + c_2 e^{-(1+i)x}$

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⑥ problem 8.5.33: a particle moves along the x-axis subject to a force toward the origin ($F = -kx$). show that the particle executes simple harmonic motion.

$F = ma \Rightarrow F = m\dfrac{d^2x}{dt^2} \Rightarrow -kx = mx'' \Rightarrow \boxed{x'' + \dfrac{k}{m}x = 0}$

let $w^2 = \dfrac{k}{m} \Rightarrow x'' + w^2 x = 0$, the auxiliary equation is

$r^2 + w^2 = 0 \Rightarrow r = \pm iw \Rightarrow r_1 = iw, \; r_2 = -iw$

$\Rightarrow x(t) = c_1 e^{iwt} + c_2 e^{-iwt} = c_3 \cos wt + c_4 \sin wt \overset{or}{=} c \sin(wt + \delta)$

This represent a periodic motion (simple harmonic).

$K = \frac{1}{2}mv^2 = \frac{1}{2}m(x')^2 = \frac{1}{2}m w^2 c^2 \cos^2(wt+\delta)$, $\bar{K} = \frac{1}{2}mw^2c^2(1/2)$

$P = \frac{1}{2}kx^2 = \frac{1}{2}(mw^2)c^2\sin^2(wt+\delta)$, $\bar{P} = \frac{1}{2}mw^2c^2(1/2)$

} time-averages of K and P

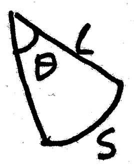
$E = K + P = \frac{1}{2}mw^2c^2\cos^2(wt+\delta) + \frac{1}{2}mw^2c^2\sin^2(wt+\delta) = \frac{1}{2}mw^2c^2 = \text{Const}$

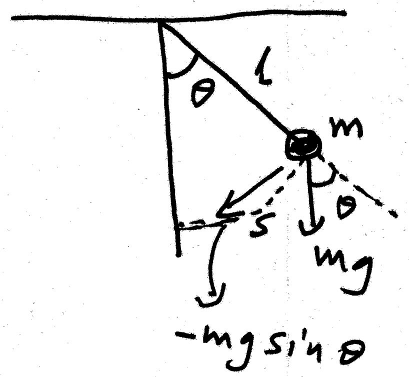
Note that the time-average of K and P is

$\bar{K} = \bar{P} = \frac{1}{2}E$

⑦ **problem 8.5.34!** Find the equation of motion of simple pendulum, given that at $t=0$, $\theta=\theta_0$ and $\theta'=0$

$$F = ma = m\frac{d^2s}{dt^2} \; ; \qquad s = l\theta, \quad s' = l\theta', \quad s'' = l\theta''$$





$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \omega^2\sin\theta = 0$$

when $\theta \ll 1$ (very small) $\Rightarrow \sin\theta \approx \theta \Rightarrow$ $\boxed{\frac{d^2\theta}{dt^2} + \omega^2\theta = 0}$

This is again simple harmonic motion

i.e $\theta'' + \omega^2\theta = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm i\omega$

$\Rightarrow \theta(t) = A\cos\omega t + B\sin\omega t$; apply B.Cs

at $t=0$, $\theta = \theta_0 \Rightarrow \theta_0 = A + 0 \Rightarrow A = \theta_0$, and

$\theta'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$, Now $\theta'(0) = 0$

$0 = 0 + B\omega \Rightarrow B = 0$

$\Rightarrow \theta(t) = A\cos\omega t = \theta_0\cos\omega t$; $\omega = \sqrt{\frac{g}{l}}$

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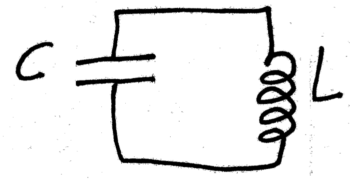
⑧ **problem 8.5.36!** Find frequency of oscillations of the circuit in Figure (1.1), when $V=0$, $R=0$



$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V$; but $V=0$, $R=0$

$\Rightarrow L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0 \Rightarrow \frac{d^2I}{dt^2} + \frac{1}{LC}I = 0 \Rightarrow I'' + \omega^2 I = 0$

where $\omega = \frac{1}{\sqrt{LC}}$

$\therefore$ $\boxed{I'' + \omega^2 I = 0}$ with $\omega = \frac{1}{\sqrt{LC}}$, when you tune the radio, you are adjusting C and/or L to make ω equal to that of the radio station

⑨ <u>problem 8.5.38:</u> solve the RLC circuit equation [5.33 or 5.34] in textbook, with $V=0$ and write down the conditions and solutions for overdamped, critically damped, and underdamped electrical oscillations.

from equation (1.2) in textbook, $L\frac{dI}{dt} + RI + \frac{1}{C}q = 0$, with $I = \frac{dq}{dt}$, we get $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$ ---- (5.33)

Now differentiating the equation ($L\frac{dI}{dt} + RI + \frac{1}{C}q = 0$) with respect to t, we get $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\underbrace{\frac{dq}{dt}}_{I} = 0$

$$\Rightarrow L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$

$$\Rightarrow \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0 \Rightarrow I'' + 2bI' + w_0^2 I = 0 \quad \text{--(5.34)}$$

$2b = \frac{R}{L}$, $w_0^2 = \frac{1}{LC}$

let us solve the last equation (5.34)

auxoliary equation reads $r^2 + 2br + w_0^2 = 0 \Rightarrow r = \frac{-2b \pm \sqrt{4b^2 - 4w_0^2}}{2}$

$$\Rightarrow r = -b \pm \sqrt{b^2 - w_0^2}$$

a) overdamped: $b^2 > w_0^2$, roots are real

$r_1 = -b + \sqrt{b^2 - w_0^2}$; $r_2 = -b - \sqrt{b^2 - w_0^2}$ $\Rightarrow I(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= -(b - \sqrt{b^2 - w_0^2})$; $= -(b + \sqrt{b^2 - w_0^2})$; $= c_1 e^{-\lambda t} + c_2 e^{-\mu t}$

$= -\lambda$; $= -\mu$

b) critically damped: $b^2 = w_0^2 \Rightarrow r = -b$, repeated roots $r_1 = r_2 = -b$

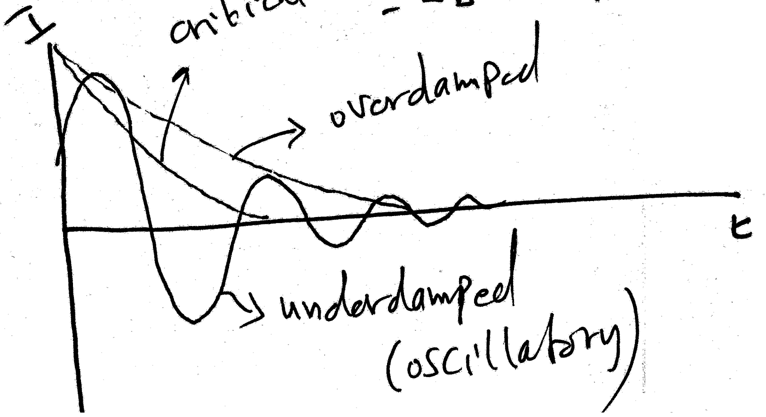
$$x(t) = (c_3 + c_4 t)e^{-bt}$$

c) underdamped: $b^2 < w_0^2$, $\sqrt{b^2 - w_0^2}$ is imaginary $\Rightarrow r = -b \pm i\sqrt{w_0^2 - b^2}$
$= -b \pm i\beta$

$$x(t) = e^{-bt}(c_5 e^{i\beta t} + c_6 e^{-i\beta t})$$

or

$$= e^{-bt}(c_7 \cos\beta t + c_8 \sin\beta t)$$



⑩ problem 8.6.1: solve $y'' - 4y = 10$

y_c: $y'' - 4y = 0 \Rightarrow r^2 - 4 = 0 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

$\Rightarrow y_c = c_1 e^{2x} + c_2 e^{-2x}$

y_p: let $y_p = c$, $y_p' = 0$, $y_p'' = 0$, substitute

$y_p'' - 4y_p = 10 \Rightarrow -4c = 10 \Rightarrow c = -\frac{10}{4} = -5/2$

$\Rightarrow y_p = -5/2 \Rightarrow$ $\boxed{y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - 5/2}$

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⑪ problem 8.6.2: solve $(D-2)^2 y = 16$

$\Rightarrow (D^2 - 4D + 4)y = 16 \Rightarrow y'' - 4y' + 4y = 16$

y_c: $y'' - 4y' + 4y = 0 \Rightarrow r^2 - 4r + 4 = 0$, $(r-2)(r-2) = 0 \Rightarrow r = 2$, repeated roots

$\Rightarrow y_c = (c_1 x + c_2) e^{2x}$

y_p: let $y_p = c \Rightarrow y_p' = y_p'' = 0 \Rightarrow$ substitute in

$y_p'' - 4y_p' + 4y_p = 16 \Rightarrow 4c = 16 \Rightarrow c = 4 \Rightarrow y_p = 4$

$\Rightarrow$ $\boxed{y = y_c + y_p = (c_1 x + c_2) e^{2x} + 4}$

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⑫ problem 8.6.3: solve $y'' + y' - 2y = e^{2x}$

y_c: $y'' + y' - 2y = 0 \Rightarrow r^2 + r - 2 = 0$, $(r-1)(r+2) = 0 \Rightarrow r_1 = 1, r_2 = -2$

$\Rightarrow y_c = c_1 e^{x} + c_2 e^{-2x}$

y_p: let $y_p = ce^{2x} \Rightarrow y_p' = 2ce^{2x}$, $y_p'' = 4ce^{2x}$, substitute

$y_p'' + y_p' - 2y_p = e^{2x} \Rightarrow 4ce^{2x} + 2ce^{2x} - 2ce^{2x} = e^{2x}$

$4c = 1 \Rightarrow c = 1/4 \Rightarrow y_p = \frac{1}{4} e^{2x}$

$\Rightarrow$ $\boxed{y = y_c + y_p = c_1 e^{x} + c_2 e^{-2x} + 1/4\, e^{2x}}$

(13) **problem 8.6.8:** solve $y'' - 16y = 40e^{4x}$

y_c : $y'' - 16y = 0 \Rightarrow r^2 - 16 = 0 \Rightarrow r^2 = 16 \Rightarrow r = \pm 2$

$y_c = c_1 e^{4x} + c_2 e^{-4x}$

y_p : let $y_p = cxe^{4x}$, $y_p' = 4cxe^{4x} + ce^{4x}$

$y_p'' = 4ce^{4x} + 16cxe^{4x} + 4ce^{4x}$

$\Rightarrow y_p'' - 16y_p = 40e^{4x}$

$\Rightarrow 4ce^{4x} + 16xce^{4x} + 4ce^{4x} - 16xce^{4x} = 40e^{4x} \Rightarrow 8c = 40$

$c = 5$

$\Rightarrow y_p = 5xe^{4x}$

$\Rightarrow y = y_c + y_p = c_1 e^{4x} + c_2 e^{-4x} + 5xe^{4x}$

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(14) **problem 8.6.10:** solve $(D-3)^2 y = 6e^{3x}$

$\Rightarrow (D^2 - 6D + 9)y = 6e^{3x} \Rightarrow y'' - 6y' + 9y = 6e^{3x}$

y_c : $y'' - 6y' + 9y = 0 \Rightarrow r^2 - 6r + 9 = 0 \Rightarrow r = 3$, repeated roots

$(r-3)(r-3) = 0$

$\Rightarrow y_c = (c_1 x + c_2)e^{3x}$

y_p : let $y_p = cx^2 e^{3x}$, $y_p' = 3cx^2 e^{3x} + 2cxe^{3x}$

$y_p'' = 9cx^2 e^{3x} + 12cxe^{3x} + 2ce^{3x}$

substitute

$y_p'' - 6y_p' + 9y_p = 6e^{3x}$

$9cx^2 e^{3x} + 12cxe^{3x} + 2ce^{3x} - 18cx^2 e^{3x} - 12cxe^{3x} + 9cx^2 e^{3x} = 6e^{3x}$

$2ce^{3x} = 6e^{3x} \Rightarrow c = 3 \Rightarrow y_p = 3x^2 e^{3x}$

$\Rightarrow y = y_c + y_p = (c_1 x + c_2)e^{3x} + 3x^2 e^{3x}$

(15) <u>problem 8.6.11</u> : solve $y'' + 2y' + 10y = 100\cos 4x$

<u>y_c</u>: $y'' + 2y' + 10y = 0 \Rightarrow r^2 + 2r + 10 = 0 \Rightarrow r = -1 \pm 3i$ (with $\alpha = -1$, $\beta = 3$)

$y_c = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$

$= e^{-x}(c_1 e^{3ix} + c_2 e^{-3ix}) = e^{-x}(c_3 \cos 3x + c_4 \sin 3x)$

<u>y_p</u>: let us solve $y'' + 2y' + 10y = 100 e^{4ix}$ and take real part

let $y_p = c e^{4ix}$, $y_p' = 4ci e^{4ix}$, $y_p'' = -16c e^{4ix}$, substitute

$y_p'' + 2y_p' + 10y_p = 100 e^{4ix}$

$\Rightarrow -16c e^{4ix} + 8ic e^{4ix} + 10c e^{4ix} = 100 e^{4ix} \Rightarrow -6c + 8ic = 100$

$\Rightarrow c = \frac{100}{-6+8i} = \frac{100}{-6+8i} \times \frac{-6-8i}{-6-8i} = -6 - 8i$

$\Rightarrow y_p = c e^{4ix} = (-6-8i)e^{4ix} = (-6-8i)(\cos 4x + i \sin 4x)$

$= \underbrace{(-6\cos 4x + 8\sin 4x)}_{\text{real part}} - i(6\sin 4x + 8\cos 4x)$

$\therefore y_p = -6\cos 4x + 8\sin 4x$

$\Rightarrow y = y_c + y_p$

$y = e^{-x}(c_3 \cos 3x + c_4 \sin 3x) + 8\sin 4x - 6\cos 4x$

- Note if the equation is $y'' + 2y' + 10y = 100 \sin 4x$, we take imaginary part of y_p i.e

$y_p = -6\sin 4x - 8\cos 4x$, so

$y = y_c + y_p = e^{-x}(c_3 \cos 3x + c_4 \sin 3x) - 6\sin 4x - 8\cos 4x$

(16) **problem 8.6.21:** Solve $5y''+6y'+2y = x^2+6x$

y_c: $5y''+6y'+2y=0 \Rightarrow 5r^2+6r+2=0 \Rightarrow r = \frac{-3}{5} \pm \frac{1}{5}i$

$\alpha = \frac{-3}{5}$, $\beta = \frac{1}{5}$

$y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$= e^{-\frac{3x}{5}}(C_1 \cos\frac{x}{5} + C_2 \sin\frac{x}{5})$

y_p: let $y_p = Ax^2+Bx+C$, $y_p' = 2Ax+B$, $y_p'' = 2A$, substitute

$5y_p''+6y_p'+2y_p = x^2+6x$

$\Rightarrow 2Ax^2 + (12A+2B)x + (10A+6B+2C) = x^2+6x$

$\Rightarrow 2A=1 \Rightarrow A = 1/2$, $12A+2B = 6 \Rightarrow B=0$, $10A+6B+2C=0 \Rightarrow C = -5/2$

$\Rightarrow y_p = \frac{1}{2}x^2 - \frac{5}{2} = \frac{1}{2}(x^2-5)$

$\Rightarrow \boxed{y = y_c + y_p = e^{-\frac{3x}{5}}(C_1 \cos\frac{x}{5} + C_2 \sin\frac{x}{5}) + \frac{1}{2}(x^2-5)}$

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(17) **problem 8.6.26:** Solve $(D-3)(D+1)y = 16x^2e^{-x} \Rightarrow (D^2-2D-3)y = 16x^2e^{-x}$

$\Rightarrow y''-2y'-3y = 16x^2e^{-x}$,

y_c: $y''-2y'-3y=0 \Rightarrow r^2-2r-3=0 \Rightarrow (r-3)(r+1)=0 \Rightarrow r_1=3, r_2=-1$

$\Rightarrow y_c = c_1e^{3x}+c_2e^{-x}$ $\quad \alpha=-1$, which is $= r_2$

y_p: let $y_p = (Ax^2+Bx+C)xe^{-x}$, find y_p' and y_p'' and substitute in $y_p''-2y_p'-3y_p = 16x^2e^{-x}$. equate coefficients and find

$A = -4/3$, $B=-1$, $C=-1/2$

$\Rightarrow y_p = -(\frac{4}{3}x^2+x+1/2)xe^{-x} = -(\frac{4}{3}x^3+x^2+1/2 x)e^{-x}$

$\Rightarrow \boxed{y = y_c + y_p = c_1e^{3x}+c_2e^{-x} - (\frac{4}{3}x^3+x^2+\frac{x}{2})e^{-x}}$

(18) problem 8.6.34 solve $y''-5y'+6y = 2e^{x} + 6x - 5$

$2e^x \downarrow y_{p_1}$, $6x \downarrow y_{p_2}$, $-5 \downarrow y_{p_3}$

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$

y_c: $y''-5y'+6y=0$

$\Rightarrow r^2-5r+6=0$

$(r-3)(r-2)=0 \Rightarrow r_1=3,\ r_2=2$

$\Rightarrow y_c = c_1e^{3x} + c_2e^{2x}$

y_p: Need to find $y_{p_1}, y_{p_2}, y_{p_3}$

y_{p_1}: solve $y''-5y'+6y = 2e^x$, let $y_{p_1} = ce^x$

$y_{p_1}' = ce^x$

$y_{p_1}'' = ce^x$

$\Rightarrow y_{p_1}'' - 5y_{p_1}' + 6y_{p_1} = 2e^x$

$ce^x - 5ce^x + 6e^x = 2e^x$

$\Rightarrow c-5+6c = 2 \Rightarrow 7c=7 \Rightarrow c=1$

$\Rightarrow \boxed{y_{p_1} = e^x}$

y_{p_2}: solve $y''-5y'+6y = 6x$, let $y_{p_2} = Ax+B$

$y_{p_2}' = A$

$y_{p_2}'' = 0$

$\Rightarrow y_{p_2}'' - 5y_{p_2}' + 6y_{p_1} = 6x$

$6Ax + 6B - 5A = 6x \Rightarrow A=1,\ B = 5/6$

$\Rightarrow y_{p_2} = x + 5/6$

y_{p_3}: solve $y''-5y'+6y = -5$, let $y_{p_3} = c$

$y_{p_3}' = 0$

$y_{p_3}'' = 0$

$\Rightarrow y_{p_3}'' - 5y_{p_3}' + 6y_{p_3} = -5$

$6c = -5 \Rightarrow c = -5/6$

$\Rightarrow \boxed{y_{p_3} = -5/6}$

$\Rightarrow y_p = y_{p_1} + y_{p_2} + y_{p_3} = e^x + x + \frac{5}{6} - \frac{5}{6} = e^x + x$

$\Rightarrow \boxed{y = y_c + y_p = c_1e^{3x} + c_2e^{2x} + e^x + x}$

(19) <u>problem 8.6.39:</u> solve equation (1.2) in textbook with $V = V_0 \sin wt$

$L \frac{dI}{dt} + RI + \frac{1}{C} q = V_0 \sin wt$, differentiate with respect to t

$L \frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt} = V_0 w \cos wt$, divide by L

$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}\frac{dq}{dt} = \frac{V_0 w}{L}\cos wt$, but $\frac{dq}{dt} = I$

$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{V_0 w}{L} \cos wt$, let $\frac{R}{L} = 2b$, $\frac{1}{LC} = w_0^2$ and $\frac{V_0 w}{L} = F$

$$\boxed{I'' + 2bI' + w_0^2 I = F \cos wt} \quad \text{---(1)}$$

we already found the complementary solution (y_c) in problem (9) (8.5.38). Now let us find the particular solution (y_p)

- let us rewrite the equation as $I'' + 2bI' + w_0^2 I = Fe^{iwt}$. once we find the solution of the last equation, we take the real part of the solution.

let $y_p = Ce^{iwt}$, $y_p' = iwCe^{iwt}$, $y_p'' = -w^2 Ce^{iwt}$

substitute in (1), where $\boxed{y_p = I}$

$y_p'' + 2b y_p' + w_0^2 y_p = Fe^{iwt}$

$-w^2 C\cancel{e^{iwt}} + 2ibwC\cancel{e^{iwt}} + w_0^2 C\cancel{e^{iwt}} = F\cancel{e^{iwt}}$

$-w^2 C + 2ibwC + w_0^2 C = F \Rightarrow C(w_0^2 - w^2) + 2ibCw = F$

$\Rightarrow C = \frac{F}{(w_0^2 - w^2) + 2ibw} = \frac{F}{(w_0^2 - w^2) + 2ibw} \times \frac{(w_0^2 - w^2) - 2ibw}{(w_0^2 - w^2) - 2ibw}$

$= \frac{F[(w_0^2 - w^2) - 2ibw]}{(w_0^2 - w^2)^2 + 4b^2 w^2}$

$$\Rightarrow y_p = I = ce^{i\omega t} = c\,[\cos\omega t + i\sin\omega t]$$

$$= \frac{F[(\omega_0^2-\omega^2)-2ib\omega]}{(\omega_0^2-\omega^2)^2+4b^2\omega^2}\,[\cos\omega t + i\sin\omega t]$$

taking the real part of the solution, we get

$$y_p = I = \frac{F(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4b^2\omega^2}\cos\omega t + \frac{2Fb\omega}{(\omega_0^2-\omega^2)^2+4b^2\omega^2}\sin\omega t$$

$$= \frac{F}{(\omega_0^2-\omega^2)+4b^2\omega^2}\left[(\omega_0^2-\omega^2)\cos\omega t + 2b\omega\sin\omega t\right]$$

Note that the current has maximum amplitude when $\omega=\omega_0$; i.e when the driving frequency equals the natural frequency ω_0, so at $\omega=\omega_0$, we have (Resonance),

$$I = \frac{F}{4b^2\omega^2}\,2b\omega\sin\omega t = \frac{F}{2b\omega}\sin\omega t$$

$$= \frac{\frac{V_0\omega}{L}}{\frac{R}{L}\omega}\sin\omega t = \frac{V_0}{R}\sin\omega t$$

we see that the maximum current is in phase with the applied voltage ($V_0\sin\omega t$). y_p is called the steady-state solution, because as t increases the complementary solution (y_c) dies out and becomes negligible. The complementary solution (transient) tends to zero very rapidly and the steady-state solution (y_p) becomes essentially the whole solution

$$y = y_c + y_p \approx y_p$$

(20) **problem 8.7.2:** solve $y'' + 2xy' = 0$,

see that y is missing $\Rightarrow$ let $y' = p$, $y'' = p'$

$\Rightarrow p' + 2xp = 0 \Rightarrow \mu = e^{\int 2x\,dx} = e^{x^2}$,

$\Rightarrow e^{x^2}[p' + 2xp] = 0 \Rightarrow e^{x^2}p' + 2xpe^{x^2} = 0 \Rightarrow [e^{x^2}p]' = 0$

integrate $e^{x^2}p = c_1 \Rightarrow p = c_1 e^{-x^2}$; but $p = y' = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = c_1 e^{-x^2} \Rightarrow dy = c_1 e^{-x^2}dx$, integrate

$y(x) = c_1 \int e^{-x^2}dx + c_2 = c_3\,\mathrm{erf}(x) + c_2$, where

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int e^{-x^2}dx$ is called the error function that will be discussed in chapter 11 (sec 9)

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(21) **problem 8.7.3:** solve $2yy'' = y'^2$, x - is missing

let $y' = p$, $y'' = pp'$ $\Rightarrow$ substitute $2ypp' = p^2$, divide by $2yp$

$\Rightarrow p' = \frac{p}{2y} \Rightarrow p' - \frac{1}{2y}p = 0$, $\Rightarrow \mu = e^{-\frac{1}{2}\int \frac{1}{y}dy} = e^{-\frac{1}{2}\ln y} = y^{-1/2}$

$\swarrow$ multiply by $y^{-1/2}$

$y^{-1/2}p' - \frac{1}{2}\frac{y^{-1/2}}{y}p = 0 \Rightarrow y^{-1/2}p - \frac{1}{2}y^{-3/2}p = 0 \Rightarrow [y^{-1/2}p]' = 0$

integrate

$y^{-1/2}p = c_1 \Rightarrow p = c_1 y^{1/2} = \frac{dy}{dx} \Rightarrow c_1 dx = y^{-1/2}dy$

integrate $c_1 x = \frac{y^{1/2}}{1/2} + c_2 = 2y^{1/2} + c_2 \Rightarrow 2y^{1/2} = c_1 x - c_2$

$\Rightarrow$ square both sides $\Rightarrow 4y = (c_1 x - c_2)^2 \Rightarrow y = \frac{1}{4}(c_1 x - c_2)$

or $y = \frac{c_1}{4}\left(x - \frac{c_2}{c_1}\right)^2$; let $\frac{c_1}{4} = a$, and $-\frac{c_2}{c_1} = b$

$= a(x+b)^2$.

note that $y = \text{const} = c$ is another solution that satisfy the differential of "$2yy'' = y'^2$"

(22) problem 8.7.17: solve $x^2y'' + xy' - 16y = 8x^4$

This is an Euler equation,

let $x = e^z \Rightarrow z = \ln x$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{-z}\frac{dy}{dz} \Rightarrow \frac{d}{dx} = e^{-z}\frac{d}{dz}, \text{ and}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = e^{-z}\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right) = e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}$$

substitute in the diff eqn, we get

$$x^2\left[e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}\right] + xe^{-z}\frac{dy}{dz} - 16y = 8x^4 \quad ; \text{ Now } x^2 = e^{2z} \text{ and } x^4 = e^{4z}$$

$$e^{2z}\left[e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}\right] + e^{z}e^{-z}\frac{dy}{dz} - 16y = 8e^{4z}$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + \frac{dy}{dz} - 16y = 8e^{4z} \Rightarrow \boxed{\frac{d^2y}{dz^2} - 16y = 8e^{4z}}$$

y_c: $r^2 - 16 = 0 \Rightarrow r = \pm 4$, $y_c = c_1e^{4z} + c_2e^{-4z}$

y_p: let $y_p = cze^{4z}$, $y_p' = 4cze^{4z} + ce^{4z}$,

$$y_p'' = 16cze^{4z} + 4ce^{4z} + 4ce^{4z} = 16cze^{4z} + 8ce^{4z}$$

$$\Rightarrow y_p'' - 16y_p = 8e^{4z} \Rightarrow 16cze^{4z} + 8ce^{4z} - 16cze^{4z} = 8e^{4z}$$

$\Rightarrow 8c = 8$

$\Rightarrow c = 1$

$\Rightarrow y_p = ze^{4z}$

$\Rightarrow y = y_c + y_p = c_1e^{4z} + c_2e^{-4z} + ze^{4z}$, using $e^z = x$, $e^{4z} = x^4$, $z = \ln x$

$$\boxed{y = c_1x^4 + c_2x^{-4} + x^4\ln x}$$

(23) problem 8.7.27: solve $xy'' - 2(x+1)y' + (x+2)y = 0$

the first solution is given $u = e^x$. Find the second solution v.

let $y = uv = e^x v$; $y' = e^x v' + ve^x$,

$y'' = e^x v' + e^x v'' + v'e^x + ve^x = e^x v'' + 2e^x v' + ve^x$

substitute into the differential equation,

$x(e^x v'' + 2e^x v' + ve^x) - 2(x+1)(e^x v' + ve^x) + (x+2)e^x v = 0$

$xe^x v'' + 2xe^x v' + xve^x - 2xe^x v' - 2xve^x - 2e^x v' - 2ve^x + xe^x v + 2e^x v = 0$

$\Rightarrow xe^x v'' - 2e^x v' = 0$, divide by xe^x

$v'' - \frac{2}{x} v' = 0 \quad \Rightarrow v'' = \frac{dv'}{dx}$

$\Rightarrow \frac{dv'}{dx} = \frac{2}{x} v' \Rightarrow \frac{dv'}{v'} = \frac{2}{x} dx$, integrate

$\ln v' = 2\ln x + \ln A = \ln x^2 A \Rightarrow v' = Ax^2$, integrate

$v = \frac{1}{3} Ax^3 + B$

$\therefore y = uv = e^x(\frac{1}{3} Ax^3 + B)$

$= \underbrace{\frac{1}{3} A x^3 e^x}_{\text{second solution}} + \underbrace{Be^x}_{\text{first solution}}$

This is the most general solution