Mathematical physics (1)
HW #1 – solution
Dr. Gassem Alzoubi

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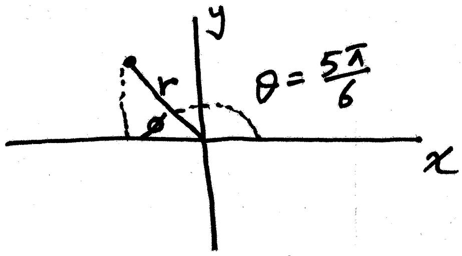
① <u>problem 2.4.4:</u> consider $z = -\sqrt{3} + i$

write z in polar form and plot it in the complex plane

$x = \mathrm{Re}(z) = -\sqrt{3}$; $y = \mathrm{Im}(z) = 1$, $r = \sqrt{x^2+y^2} = \sqrt{(-\sqrt{3})^2+1^2}$
$= \sqrt{4} = 2$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = -30^\circ = -\frac{\pi}{6}$



$\therefore\ \theta = \pi - \frac{\pi}{6} = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}$ with $+x$

$\therefore\ z = -\sqrt{3} + i = re^{i\theta} = 2e^{i\frac{5\pi}{6}}$

see that $|z| = r = 2$ and $\bar{z} = z^* = -\sqrt{3} - i = 2e^{-i\frac{5\pi}{6}}$

and $|\bar{z}| = 2$

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② <u>problem 2.5.6:</u> consider $z = \left(\frac{1+i}{1-i}\right)^2$; simplify to the $x+iy$ form and write it in polar form.

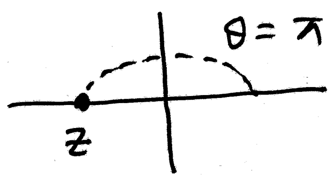
$z = \left(\frac{1+i}{1-i}\right)^2 = \left(\frac{1+i}{1-i}\right)^2 \times \left(\frac{1+i}{1+i}\right)^2 = \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^2 = \left(\frac{2i}{2}\right)^2 = (i)^2 = -1$

$\Rightarrow z = -1 \Rightarrow x = -1, y = 0 \Rightarrow r = \sqrt{x^2+y^2} = 1$, and $\theta = \tan^{-1}\left(\frac{0}{-1}\right)$
$= 0, \pi$ ✓

$\Rightarrow z = re^{i\theta} = e^{i\pi}$

– another method

write $1+i = \sqrt{2}e^{i\pi/4}$ and $1-i = \sqrt{2}e^{-i\pi/4}$



$\Rightarrow z = \left(\frac{1+i}{1-i}\right)^2 = \left(\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}}\right)^2 = \left(e^{i\pi/2}\right)^2 = e^{i\pi} = \cos\pi + i\sin\pi$ ✓
$= -1$

as expected

③ **problem 2.5.29:** Find $|(1+2i)^3|$

using $|z| = \sqrt{z\bar{z}}$, we have

$$|(1+2i)^3| = \sqrt{(1+2i)^3(1-2i)^3} = \sqrt{[(1+2i)(1-2i)]^3} = \sqrt{[1^2+2^2]^3}$$

$$= \sqrt{5^3} = \sqrt{125} = \sqrt{25\times5} = 5\sqrt{5}$$

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④

Problem 2.5.47: consider the eq^n $(x+iy)^3 = -1$

solve for all possible real values of x and y.

$(x+iy)^3 = x^3 - 3xy^2 + i(3yx^2 - y^3) = -1$ ⇒ Three solutions

$x^3 - 3xy^2 = -1$ --- (1)

and $3yx^2 - y^3 = 0$ --- (2)

one solution to eq^n (1) is $y=0$; substitute back in (1)

⇒ $x^3 = -1$ ⇒ $x=-1$; so the first solution is $(-1,0)$

- Now to find the other solutions corresponding to $y \neq 0$,

divide (2) by y, we get $3x^2 - y^2 = 0$ ⇒ $y = \pm\sqrt{3}\,x$.

Now insert this back in (1), we get $x^3 - 3x(3x^2) = -1$

⇒ $x^3 - 9x^3 = -1$ ⇒ $-8x^3 = -1$ ⇒ $x^3 = \frac{1}{8}$ ⇒ $x = \frac{1}{2}$

∴ the 2nd and 3^{d} solutions are $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{3})$

∴ $(x,y) = (-1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, -\frac{\sqrt{3}}{2})$ all solutions

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⑤ **problem 2.5.49:** consider $|1-(x+iy)| = x+iy$

solve for all possible real values of x and y

$|1-(x+iy)| = x+iy \Rightarrow |(1-x)-iy| = x+iy \Rightarrow$

$\sqrt{(1-x)^2+y^2} = x+iy \Rightarrow \sqrt{(1-x)^2+y^2} = x$ ---(1)

and $y=0$ ------- (2)

from (1) $\sqrt{(1-x)^2} = x \Rightarrow 1-x = x \Rightarrow x = 1/2$

$\therefore (x,y) = (1/2, 0)$

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⑥ problem 2.5.59: consider $|z+3i| = 4$.

describe this equation geometrically to find the set of points satisfying it in the complex plane

$|z+3i| = 4 \Rightarrow |x+iy+3i| = 4 \Rightarrow |x+i(y+3)| = 4$

$\Rightarrow \sqrt{[x+i(y+3)][x-i(y+3)]} = 4 \Rightarrow \sqrt{x^2+(y+3)^2} = 4$

$\Rightarrow x^2+(y+3)^2 = 16 = 4^2 \Rightarrow$ circle with radius $r=4$ and centered at $(0,-3)$

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⑦ problem 2.5.68: Consider $z = \cos 2t + i\sin 2t$

Find v and a and describe the motion.

first let us describe the motion

$|z| = \sqrt{z\bar{z}} = \sqrt{(\cos 2t + i\sin 2t)(\cos 2t - i\sin 2t)}$

$= \sqrt{\cos^2 2t + \sin^2 2t} = \sqrt{1} = 1$

$\therefore |z| = 1$, but $z = x+iy$

$\therefore |z| = 1 \Rightarrow \sqrt{z\bar{z}} = 1 \Rightarrow \sqrt{(x+iy)(x-iy)} = 1$

$\Rightarrow \sqrt{x^2+y^2} = 1 \quad \Rightarrow x^2+y^2 = 1$ $\therefore$ motion is around a circle with radius $r=1$ and centered at $(0,0)$

now

$$\frac{dz}{dt} = -2\sin 2t + 2i\cos 2t \; ; \; \frac{d^2z}{dt^2} = -4\cos 2t - 4i\sin 2t$$

$$\Rightarrow v = \left|\frac{dz}{dt}\right| = \sqrt{(-2\sin 2t + 2i\cos 2t)(-2\sin 2t - 2i\cos(2t))}$$

$$= \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2\underbrace{\sqrt{\sin^2 2t + \cos^2 2t}}_{1}$$

$$= 2$$

similarly

$$a = \left|\frac{d^2z}{dt^2}\right| = \sqrt{(-4\cos 2t - 4i\sin 2t)(-4\cos 2t + 4i\sin(2t))}$$

$$= \sqrt{16\cos^2 2t + 16\sin^2 2t} = 4\sqrt{\cos^2 2t + \sin^2 2t}$$

$$= 4$$

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8) problem 2.9.12 write $4e^{-i\frac{8\pi}{3}}$ in $x+iy$ form

$$4e^{-i\frac{8\pi}{3}} = 4\left[\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right]; \quad \frac{8\pi}{3} = \frac{6\pi}{3} + \frac{2\pi}{3} = 2\pi + \underbrace{\frac{2\pi}{3}}$$

$$= 4\left[\cos\underbrace{\frac{2\pi}{3}}_{120^\circ} - i\sin\frac{2\pi}{3}\right] = 4\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = -2 - 2\sqrt{3}\,i$$

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9) problem 2.9.26 write $\left(\frac{2i}{i+\sqrt{3}}\right)^{19}$ in $x+iy$ form

$2i$: $r = \sqrt{x^2+y^2} = \sqrt{4} = 2$, $\theta = \tan^{-1}\left(\frac{2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$\therefore 2i = 2e^{i\pi/2}$

and $i+\sqrt{3}$: $r=\sqrt{x^2+y^2}=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{4}=2$

$\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\pi/6$

$\Rightarrow i+\sqrt{3}=2e^{i\pi/6}$

$\Rightarrow \left(\frac{2i}{\sqrt{3}+i}\right)^{19}=\left(\frac{\cancel{2}e^{i\pi/2}}{\cancel{2}e^{i\pi/6}}\right)^{19}=\frac{e^{i\frac{19\pi}{2}}}{e^{i\frac{19\pi}{6}}}=e^{i\left(\frac{19\pi}{2}-\frac{19\pi}{6}\right)}$

$=e^{i\frac{38\pi}{6}}=\cos\frac{38\pi}{6}+i\sin\frac{38\pi}{6}$

but $\frac{38\pi}{6}=\frac{36\pi}{6}+\frac{2\pi}{6}=6\pi+\frac{\pi}{3}$ ✓

$=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}=\frac{1}{2}(1+i\sqrt{3})$

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(10) problem 2.9.27

show that for any real y, $|e^{iy}|=1$, and show that $|e^{z}|=e^{x}$

$\Rightarrow |e^{iy}|=\sqrt{e^{iy}e^{-iy}}=\sqrt{1}=1$ ✓

$|e^{z}|=|e^{x+iy}|=\sqrt{e^{x+iy}\,e^{x-iy}}$

$=\sqrt{e^{2x}}=\sqrt{(e^{x})^{\cancel{2}}}=e^{x}$ ✓

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(11) problem 2.9.28 show that $|z_1z_2|=|z_1||z_2|$

$|z_1z_2|=|r_1e^{i\theta_1}r_2e^{i\theta_2}|=|r_1r_2e^{i(\theta_1+\theta_2)}|=r_1r_2\underbrace{|e^{i(\theta_1+\theta_2)}|}_{1}$

$=r_1r_2$

$=|z_1||z_2|$

but $r_1=\text{mod } z_1=|z_1|$

and $r_2=\text{mod } z_2=|z_2|$

similarly $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

(12) problem 2.9.33: evaluate $|2e^{3+i\pi}|$

$$|2e^{3+i\pi}| = |2e^3 e^{i\pi}| = 2e^3 \underbrace{|e^{i\pi}|}_{1} = 2e^3$$

(13) problem 2.9.35 evaluate $|3e^{5i} \cdot 7e^{-2i}|$

$$|3e^{5i} \cdot 7e^{-2i}| = |21 e^{3i}| = 21 \underbrace{|e^{3i}|}_{1} = 21$$

(14) problem 2.9.38 evaluate $\left|\frac{e^{i\pi}}{1+i}\right|$

$$\left|\frac{e^{i\pi}}{1+i}\right| = \frac{|e^{i\pi}|}{|1+i|} = \frac{1}{\sqrt{(1+i)(1-i)}} = \frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$$