Phys 741
Statistical Mechanics
Special Integrals, Functions, and Series Sheet

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1. The n-dimensional volume and surface area of a hypersphere of radius R is $V_n(R) = \frac{\pi^{n/2}R^n}{\Gamma(\frac{n}{2}+1)} = \frac{\pi^{n/2}R^n}{(n/2)!}$ and $S_n(R) = \frac{n\pi^{n/2}R^{n-1}}{\Gamma(\frac{n}{2}+1)} = \frac{2\pi^{n/2}R^{n-1}}{\Gamma(\frac{n}{2})}$, where $\Gamma(\frac{n}{2}+1) = (n/2)!$ and $\Gamma(\frac{n}{2}+1) = \frac{n}{2}\Gamma(n/2)$
2. Stirlings formula $\ln n! \approx n\ln n - n$
3. The Gamma integrals : $\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$, $\Gamma(n) = (n-1)!$, $\Gamma(n+1) = n\Gamma(n)$, $\int_0^\infty x^n\, e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$

n	$-3/2$	-1/2	1/2	1	3/2	2	5/2	3	4
$\Gamma(n)$	$\frac{4\sqrt{\pi}}{3}$	$-2\sqrt{\pi}$	$\sqrt{\pi}$	1	$\frac{\sqrt{\pi}}{2}$	1	$\frac{3\sqrt{\pi}}{4}$	2	6

4. The Gaussian integrals: $\int_{-\infty}^\infty e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$, $\int_{-\infty}^\infty x^2 e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$, $\int_{-\infty}^\infty x^4 e^{-\alpha x^2}dx = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$, a general formula when the integrand is even is given by $\int_{-\infty}^\infty x^{2n} e^{-\alpha x^2}dx = \frac{(2n)!}{n!2^{2n}}\sqrt{\frac{\pi}{\alpha^{2n+1}}}$, note that because the integrand is even we can take $\int_{-\infty}^\infty x^{2n} e^{-\alpha x^2}dx = 2\int_0^\infty x^{2n} e^{-\alpha x^2}dx$

   $\int_0^\infty x e^{-\alpha x^2}dx = \frac{1}{2\alpha}$, $\int_0^\infty x^3 e^{-\alpha x^2}dx = \frac{1}{2\alpha^2}$, $\int_0^\infty x^5 e^{-\alpha x^2}dx = \frac{1}{\alpha^3}$, a general formula when the integrand is odd and from 0 to ∞ is given by $\int_0^\infty x^{2n+1} e^{-\alpha x^2}dx = \frac{n!}{2\alpha^{n+1}}$, note that because the integrand is odd , $\int_{-\infty}^\infty x^{2n+1} e^{-\alpha x^2}dx = 0$
5. Riemann zeta function: $\zeta(p) = \sum_{j=1}^\infty \frac{1}{j^p}$

p	1	3/2	2	5/2	3	4	5	6
$\zeta(p)$	∞	2.612	$\frac{\pi^2}{6} \approx 1.645$	1.341	1.20206	$\frac{\pi^4}{90} \approx 1.0823$	1.0369	$\frac{\pi^6}{945} \approx 1.017$

6. The Bose integrals : $g_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x - 1}$, $\int_0^\infty \frac{x^n dx}{e^x-1} = \Gamma(n+1)\,\zeta(n+1)$, $\int_0^\infty \frac{x^n e^x dx}{(e^x-1)^2} = n\Gamma(n)\,\zeta(n)$,
7. The Fermi integrals : $f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x + 1}$, $\int_0^\infty \frac{x^n dx}{e^x+1} = (1 - \frac{1}{2^n})\,\Gamma(n+1)\,\zeta(n+1)$
8. Sommerfield expansion formula : $\int_0^\infty \frac{F(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu}+1} = \int_0^\mu F(\varepsilon)d\varepsilon + \frac{\pi^2}{6}(k_BT)^2F'(\mu)$
9. Infinite geometric series: $\sum_{n=0}^\infty x^n = \frac{1}{1-x}$, $\sum_{n=0}^\infty nx^n = \frac{x}{(1-x)^2}$, $\sum_{n=0}^\infty n^2x^n = \frac{x(x+1)}{(1-x)^3}$
10. Finite geometric series: $a\sum_{n=0}^N x^n = a\ \frac{1-x^n}{1-x}$
11. Maclaurin series (power series): $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, $(1+x)^n = \sum_{k=0}^n \binom{n}{k}x^k$, and $(a+x)^n = \sum_{k=0}^n \binom{n}{k}x^k a^{n-k}$,

    where the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Good Luck