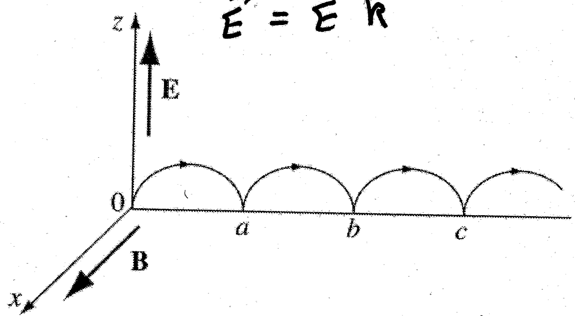
Example 5.2. Cycloid Motion. A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction, as shown in figure. positive charge (q) is released from the origin; what path will it follow?

$\vec{B} = B\hat{i}$
$\vec{E} = E\hat{k}$

Let's think it through qualitatively, first. Initially, the particle is at rest, so the magnetic force is zero, and the electric field accelerates the charge in the z-direction. As it picks up speed, a magnetic force develops which pulls the charge around to the right. The faster it goes, the stronger F_{mag} becomes; eventually, it curves the particle back around towards the y-axis. At this point the charge is moving against the electrical force, so it begins to slow down; the magnetic force then decreases, and the electrical force takes over, bringing the particle to rest at point *a*, in figure. There the entire process commences anew, carrying the particle over to point *b*, and so on. $\Rightarrow$ No motion along x-direction $\Rightarrow U_x = 0$



$$\vec{F} = q\vec{E} + q\,\vec{U}\times\vec{B}\ ; \qquad \vec{U}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & U_y & U_z \\ B & 0 & 0 \end{vmatrix} = BU_z\hat{j} - BU_y\hat{k}$$

$$= qE\hat{k} + qBU_z\hat{j} - qBU_y\hat{k}$$

$$= qBU_z\hat{j} + q(E - BU_y)\hat{k}$$

as seen No force along the x-direction

$$\therefore\ \vec{U} = (0, U_y, U_z) \Rightarrow m\frac{d\vec{U}}{dt} = \vec{F}$$

$$\Rightarrow m\frac{dU_y}{dt}\hat{j} + m\frac{dU_z}{dt}\hat{k} = qBU_z\hat{j} + q(E - BU_y)\hat{k}$$

$$\Rightarrow m\frac{dU_y}{dt} = qBU_z \Rightarrow \frac{dU_y}{dt} = \frac{qB}{m}U_z = \omega U_z \quad \text{---} \quad (1)$$

$$\text{and } m\frac{dU_z}{dt} = q(E - BU_y) \Rightarrow \frac{dU_z}{dt} = \frac{qE}{m} - \frac{qB}{m}U_y \quad \text{----} \quad (2)$$

(1) and (2) are two coupled differential equations, they are easily solved by differentiating the first and using the second to eliminate $\dot{U}_z$;

$\dot{v}_y = \omega v_z$ --- (1) where $\omega = \frac{qB}{m}$

$\dot{v}_z = \frac{qE}{m} - \omega v_y$ --- (2)

so differentiating (1), $\ddot{v}_y = \omega \dot{v}_z = \omega\left(\frac{qE}{m} - \omega v_y\right)$

$\Rightarrow \boxed{\ddot{v}_y + \omega^2 v_y = \frac{qE\omega}{m}}$ --- (3) non-homogeneous 2nd order linear diff. eqn

similarly differentiate (2) and use (1) to eliminate $\dot{v}_y$

$\ddot{v}_z = -\omega \dot{v}_y = -\omega(\omega v_z) = -\omega^2 v_z$

$\Rightarrow \boxed{\ddot{v}_z + \omega^2 v_z = 0}$ --- (4) homogeneous 2nd order diff eqn

the solution of (3) is composed of two parts; complementary solution and particular solution

- comp. solution $\ddot{v}_y + \omega^2 v_y = 0 \Rightarrow m^2 + \omega^2 = 0$ auxillary eqn

$m^2 = -\omega^2$

$m = \pm\sqrt{-\omega^2} = \pm\sqrt{-1}\,\omega = \pm i\omega$

$v_y^c(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

$= c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

$= c_1 \cos\omega t + i c_1 \sin\omega t + c_2 \cos\omega t - c_2 i \sin\omega t$

$= (c_1 + c_2)\cos\omega t + (c_1 - c_2) i \sin\omega t$

$v_y^c(t) = A_1 \cos\omega t + A_2 \sin\omega t$

- particular solution : since the term in the R.H.S is constant, then $v_y^p(t) = \text{constant} = D$; in general the particular solution takes the same form of what exists on the R.H.S, for example $ay'' + by' + cy = 4x^2$, then $y^p(x) = ax^2 + bx + c$, a polynomial of the same degree

$\therefore\ U_y^p(t) = D$; substitute this into (3), we get

$$\ddot{U}_y + \omega^2 U_y = \frac{qE\omega}{m} \Rightarrow 0 + \omega^2 D = \frac{qE\omega}{m} \Rightarrow D = \frac{qE\omega}{m\omega^2} = \frac{qE}{m\omega}$$

$$\Rightarrow D = \frac{\cancel{q}E}{\cancel{m}\frac{\cancel{q}B}{\cancel{m}}} = \frac{E}{B}$$

$$\therefore\ U_y(t) = U_y^c(t) + U_y^p(t)$$

$$\boxed{U_y(t) = A_1 \cos\omega t + A_2 \sin\omega t + \frac{E}{B}} \quad \text{---- (5)}$$

the solution of equation (4) is

$$\boxed{U_z(t) = B_1 \cos\omega t + B_2 \sin\omega t} \quad \text{---- (6)}$$

now differentiate (5) and equate the result with (1)

$$\dot{U}_y = -A_1\omega \sin\omega t + A_2\omega \cos\omega t = \omega U_z$$

$$\Rightarrow -A\omega \sin\omega t + A_2\omega \cos\omega t = \omega B_1 \cos\omega t + \omega B_2 \sin\omega t$$

$$\Rightarrow A_2\omega = \omega B_1 \Rightarrow A_2 = B_1$$

and $-A_1\omega = \omega B_2 \Rightarrow A_1 = -B_2$, so eq^{ns} (5) and (6) becomes

$$\Rightarrow U_y = A_1 \cos\omega t + A_2 \sin\omega t + \frac{E}{B} \quad \text{---- (7)}$$

$$U_z = A_2 \cos\omega t - A_1 \sin\omega t \quad \text{---- (8)}$$

integrating (7) and (8), we get

$$y(t) = \frac{A_1}{\omega} \sin\omega t - \frac{A_2}{\omega} \cos\omega t + \frac{E}{B} t + A_3 \quad \text{---- (9)}$$

$$z(t) = \frac{A_2}{\omega} \sin\omega t + \frac{A_1}{\omega} \cos\omega t + A_4 \quad \text{---- (10)}$$

now A_1, A_2, A_3, A_4 are found from Boundary conditions since the particle start from rest at the origin

$\Rightarrow v_y(t=0) = v_z(t=0) = 0$ and $y(t=0) = z(t=0) = 0$

so from v_y and v_z equations (7) and (8), we get

$$0 = A_1 + \frac{E}{B} \Rightarrow \boxed{A_1 = -\frac{E}{B}} \quad \text{and} \quad \boxed{0 = A_2}$$

$$\therefore y(t) = -\frac{E}{B\omega}\sin\omega t + \frac{E}{B}t + A_3 \quad \text{--- (11)}$$

$$z(t) = \frac{E}{B\omega}\cos\omega t + A_4 \quad \text{---- (12)}$$

Now from the second B.Cs ($y(t=0) = z(t=0) = 0$)

we get

$$0 = A_3 \quad \text{and} \quad 0 = -\frac{E}{B\omega} + A_4 \Rightarrow A_4 = +\frac{E}{B\omega}$$

so $y(t) = -\frac{E}{B\omega}\sin\omega t + \frac{E}{B}t = \frac{E}{B\omega}(\omega t - \sin\omega t)$ -- (13)

and $z(t) = -\frac{E}{B\omega}\cos\omega t + \frac{E}{B\omega} = \frac{E}{B\omega}(1 - \cos\omega t)$ --- (14)

define $\frac{E}{B\omega} = R$, radius of the circle, then

to find the combined equation of the path, take

$$y^2 + z^2 = R^2(\omega t - \sin\omega t)^2 + R^2(1 - \cos\omega t)^2$$

$$= R^2[\omega^2 t^2 + \sin^2\omega t - 2\omega t \sin\omega t]$$

$$+ R^2[1 + \cos^2\omega t - 2\cos\omega t]$$

$$= R^2\omega^2 t^2 + R^2\sin^2\omega t - 2\omega t R^2 \sin\omega t$$

$$- 2R^2\cos\omega t$$

$$+ R^2 + R^2\cos^2\omega t$$

using $\sin^2 wt + \cos^2 wt = 1$, we get

$$y^2 + z^2 = R^2 w^2 t^2 + 2R^2 - 2wtR^2 \sin wt - 2R^2 \cos wt \quad \text{--- (15)}$$

but from (13) and (14), we have

$$\frac{y}{R} = wt - \sin wt \Rightarrow -\sin wt = \frac{y}{R} - wt$$

and $\frac{z}{R} = 1 - \cos wt \Rightarrow -\cos wt = \frac{z}{R} - 1$ } substitute in (15)

$$\Rightarrow y^2 + z^2 = R^2 w^2 t^2 + 2R^2 + 2wtR^2\left(\frac{y}{R} - wt\right) + 2R^2\left(\frac{z}{R} - 1\right)$$

$$= R^2 w^2 t^2 + \cancel{2R^2} + 2wtRy - 2w^2t^2R^2 + 2Rz - \cancel{2R^2}$$

$$= R^2 w^2 t^2 + 2wtRy - 2w^2t^2R^2 + 2Rz$$

$$= -R^2 w^2 t^2 + 2wtRy + 2Rz$$

$$\Rightarrow \underbrace{y^2 + R^2 w^2 t^2 - 2wtRy}_{(y - Rwt)^2} + z^2 - 2Rz = 0$$

$\Rightarrow (y - Rwt)^2 + z^2 - 2Rz = 0$, completing the square of $z^2 - 2Rz$ by adding $(R^2 - R^2)$, we get

$$(y - Rwt)^2 + z^2 - 2Rz + R^2 = R^2$$

$$(y - \underset{\downarrow v}{Rwt})^2 + (z - R)^2 = R^2$$

equation of a circle of radius R, whose center $(0, Rwt, R)$ travels in the y-direction with a constant speed of $v = wR = \cancel{w} \frac{E}{B\cancel{w}} = \frac{E}{B}$. The curve of the motion is called cycloid and shown in the figure.