Electromagnetic theory (2)

HW # 16 - Solution

Dr. Gassem Alzoubi

Problem: Consider an electromagnetic wave propagating in free space in the +z direction. The magnetic field of the wave is given by

$$\vec{B}(z,t) = \frac{E_o}{c}\, e^{i(kz-\omega t)}\, \hat{y}$$

Find an expression for the electric field $\vec{E}(z,t)$

using Maxwell's relation (Ampors' law)

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \; ; \qquad \text{using} \quad \frac{\partial}{\partial t} = -i\omega$$

$$\Rightarrow \nabla \times \vec{B} = \frac{-i\omega}{c^2}\vec{E} \quad \Rightarrow \quad \vec{E} = \frac{c^2}{-i\omega}\nabla \times \vec{B}$$

Now

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{E_0}{c} e^{i(kz-\omega t)} & 0 \end{vmatrix}$$

$$= \hat{x}\left[0 - \frac{E_0}{c} ik\, e^{i(kz-\omega t)}\right]$$

$$= -i\frac{E_0 k}{c}\, e^{i(kz-\omega t)}\,\hat{x}$$

$$\Rightarrow \vec{E} = \frac{c^{\cancel{2}}}{-i\omega}(-i)\frac{E_0 k}{\cancel{c}}\, e^{i(kz-\omega t)}\,\hat{x}$$

$$= \frac{c}{\left(\frac{\omega}{k}\right)} E_0\, e^{i(kz-\omega t)}\,\hat{x} = \frac{\cancel{c}}{\cancel{c}} E_0\, e^{i(kz-\omega t)}\,\hat{x}$$

$$= E_0\, e^{i(kz-\omega t)}\,\hat{x}$$

Problem 9.9 Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to the point (1, 1, 1), with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$.

a) $\vec{k} = -k\hat{x}$; $\hat{n} = \hat{z}$; $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$; $\vec{k}\cdot\vec{r} = -kx = -\frac{\omega}{c}x$

$\Rightarrow \vec{E}(x,t) = E_0 \cos(kx + \omega t)\hat{z} = E_0 \cos(\frac{\omega}{c}x + \omega t)\hat{z}$

$\vec{B}(x,t) = \frac{1}{c}(-\hat{x}\times\vec{E}) = -\frac{E_0}{c}\cos(\frac{\omega}{c}x + \omega t)\underbrace{(\hat{x}\times\hat{z})}_{-\hat{y}}$

$= \frac{E_0}{c}\cos(\frac{\omega}{c}x + \omega t)\,\hat{y}$

b) let $\vec{k} = k\hat{d}$; $\hat{d} = \hat{x} + \hat{y} + \hat{z}$; every unit vector should be normalized to 1 as its value $|\hat{d}| = 1 \Rightarrow \hat{d} = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$

$\Rightarrow \vec{k} = \frac{\omega}{c\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$; Now let $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$

but $\hat{n}$ is parallel to the xz plane, so it has no y-component

$\Rightarrow n_y = 0 \Rightarrow \hat{n} = n_x\hat{x} + n_z\hat{z}$. in addition

$\vec{k}\cdot\hat{n} = 0 \Rightarrow \frac{\omega}{c\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})\cdot(n_x\hat{x} + n_z\hat{z}) = 0$ [same as $\hat{d}\cdot\hat{n} = 0$]

$\Rightarrow n_x + n_z = 0 \Rightarrow n_x = -n_z \Rightarrow \hat{n} = n_x\hat{x} - n_x\hat{z}$, again

$\hat{n}$ must be normalized $\Rightarrow \hat{n} = \frac{1}{\sqrt{2}}n_x(\hat{x} - \hat{z})$; $\Rightarrow |\hat{n}| = 1 \Rightarrow \frac{n_x^2}{2} + \frac{n_x^2}{2} = 1 \Rightarrow n_x^2 = 1$

$\Rightarrow n_x = \pm 1$; both values work

take $n_x = 1 \Rightarrow \hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z})$, so

$\vec{E}(\vec{r},t) = E_0\cos(\vec{k}\cdot\vec{r} - \omega t)\,\hat{n} = E_0\cos\left(\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right)\frac{\hat{x} - \hat{z}}{\sqrt{2}}$

$\vec{B}(\vec{r},t) = \frac{1}{c}(\hat{d}\times\vec{E}) = \frac{1}{c}\frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})\times E_0\cos\left(\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right)\frac{\hat{x} - \hat{z}}{\sqrt{2}}$

$= \frac{E_0}{c}\frac{1}{\sqrt{6}}\cos\left(\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right)\left[-\hat{x} + 2\hat{y} - \hat{z}\right]$; where

$(\hat{x} + \hat{y} + \hat{z})\times(\hat{x} - \hat{z}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \end{vmatrix} = -\hat{x} + 2\hat{y} - \hat{z}$

Problem: Consider an electromagnetic wave propagating in free space in the +z-direction with angular frequency $\omega = 2\pi \times 10^8$ rad/s and amplitude of $E_o = 10$ V/m. The electric field of the electromagnetic wave is polarized in the direction $\hat{n} = \hat{x} + \hat{y}$.

a) Find the frequency (v), wave umber (k), wavelength (λ)of the wave
b) Write down an expressions for the real electric and magnetic fields of the wave
c) Show that **E**, **B**, and ***k*** all normal to one another
d) Find the intensity of the wave
e) If this wave is normally incident on a surface and the wave is completely absorbed, find the pressure exerted by the wave on the surface

a) $\nu = \frac{\omega}{2\pi} = \frac{2\pi \times 10^8}{2\pi} = 1\times 10^8 \ Hz$, $k = \frac{\omega}{c} = \frac{2\pi \times 10^8}{3\times 10^8} = \frac{2\pi}{3} \ m^{-1}$,

$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\frac{2\pi}{3}} = 3 \ m$ or using $\lambda = \frac{c}{\nu} = \frac{3\times 10^8}{1\times 10^8} = 3 \ m$

b) $\vec{E} = E_o \cos(kz - \omega t)\hat{n}$, first normalize $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$

$= 10 \cos(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)$, and

$\vec{B} = \frac{1}{c}(\hat{z}\times\vec{E}) = \frac{10}{c}\cos(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left[\hat{z}\times\frac{(\hat{x}+\hat{y})}{\sqrt{2}}\right]$

$= \frac{10}{c}\cos(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)$

c) $\vec{E}\cdot\vec{k} = \frac{2\pi}{3}\cdot 10 \cos(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left[\frac{\hat{x}+\hat{y}}{\sqrt{2}}\cdot\hat{z}\right] =$ zero

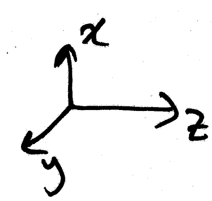
$\vec{B}\cdot\vec{k} = \frac{2\pi}{3}\cdot\frac{10}{c}\cos(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left\{\frac{\hat{y}-\hat{x}}{\sqrt{2}}\cdot\hat{z}\right\} =$ zero , and

$\vec{E}\cdot\vec{B} = 10\cdot\frac{10}{c}\cos^2(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left[\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)\cdot\left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)\right] =$ zero

$\Downarrow$

$\left[\frac{1}{\sqrt{2}}(1-1)\right] =$ zero

d) $I = \langle\vec{S}\rangle = \left\langle\frac{\vec{E}\times\vec{B}}{\mu_o}\right\rangle$



$= \frac{1}{\mu_o} 10\cdot\frac{10}{c}\cos^2(\frac{2\pi}{3}z - 2\pi\times 10^8 t)\left[\frac{\hat{x}+\hat{y}}{\sqrt{2}}\times\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right]$

$= \frac{100}{\mu_o c}\cdot\frac{1}{2}\cdot\frac{1}{2}\left[\hat{z} - (-\hat{z})\right] = \frac{100}{2\mu_o c}\hat{z} = \frac{100}{2\times 4\pi\times 10^{-7}\times 3\times 10^8}\hat{z} \approx 0.13 \frac{W}{m^2\cdot s}$

e) $P = \frac{\langle\vec{S}\rangle}{c} = \frac{0.13}{3\times 10^8} = 4.3\times 10^{-10}$ Pa , using 1 atm $= 1.03\times 10^5$ Pa

-10 -15

Problem 9.14 Calculate the exact reflection and transmission coefficients, without assuming $\mu_1 = \mu_2 = \mu_0$. Confirm that $R + T = 1$.

we already found that

$$E_{0T} = \left(\frac{2}{1+\beta}\right) E_{0I} \quad \text{and} \quad E_{0R} = \left(\frac{1-\beta}{1+\beta}\right) E_{0I}$$

Now

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \Rightarrow$$

$$R = \left(\frac{1-\beta}{1+\beta}\right)^2 \quad \text{---} \quad (1)$$

and

$$T = \frac{I_T}{I_I} = \underbrace{\frac{\epsilon_2 v_2}{\epsilon_1 v_1}}_{\beta} \left(\frac{E_{0T}}{E_{0I}}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2$$

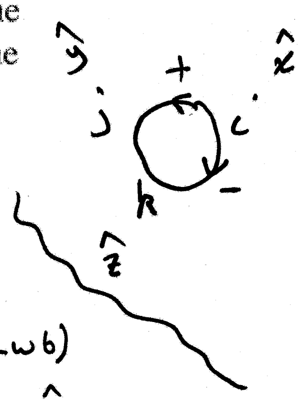
$$\Rightarrow \quad T = \beta\left(\frac{2}{1+\beta}\right)^2 \quad \text{---} \quad (2)$$

$$\Rightarrow T + R = \beta\left(\frac{2}{1+\beta}\right)^2 + \left(\frac{1-\beta}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} + \frac{(1-\beta)^2}{(1+\beta)^2}$$

$$= \frac{4\beta + (1-\beta)^2}{(1+\beta)^2} = \frac{4\beta + 1 + \beta^2 - 2\beta}{(1+\beta)^2}$$

$$= \frac{1 + 2\beta + \beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1$$

Problem 9.15 In writing Eqs. 9.76 and 9.77, I tacitly assumed that the reflected and transmitted waves have the same *polarization* as the incident wave—along the x direction. Prove that this *must* be so. [*Hint:* Let the polarization vectors of the transmitted and reflected waves be



$$\hat{\mathbf{n}}_T = \cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{y}}, \quad \hat{\mathbf{n}}_R = \cos\theta_R\,\hat{\mathbf{x}} + \sin\theta_R\,\hat{\mathbf{y}},$$

and prove from the boundary conditions that $\theta_T = \theta_R = 0$.]

incident: $\vec{\tilde{E}}_I = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\,\hat{x}$; $\vec{\tilde{B}}_I = \frac{1}{v_1}(\hat{z}\times \vec{\tilde{E}}_I)$

$$= \frac{1}{v_1}\vec{\tilde{E}}_{0I}\, e^{i(-k_1 z - \omega t)}\,\hat{y}$$

Reflected: $\vec{\tilde{E}}_R = \vec{\tilde{E}}_{0R}\, e^{i(-k_1 z - \omega t)}\,\hat{n}_R$

$$= \vec{\tilde{E}}_{0R}\, e^{i(-k_1 z - \omega t)}\,(\cos\theta_R\,\hat{x} + \sin\theta_R\,\hat{y})$$

and $\vec{B}_R = \frac{1}{v_1}(-\hat{z}\times\vec{\tilde{E}}_R) = \frac{-1}{v_1}\vec{\tilde{E}}_{0R}\, e^{i(-k_1 z - \omega t)}\,(\cos\theta_R\,\hat{y} - \sin\theta_R\,\hat{x})$

Transmitted: $\vec{\tilde{E}}_T = \vec{\tilde{E}}_{0T}\, e^{i(k_2 z - \omega t)}\,\hat{n}_T$

$$\vec{B}_T = \frac{1}{v_2}(\hat{z}\times\vec{\tilde{E}}_{0T}) = \frac{1}{v_2}\vec{\tilde{E}}_{0T}\, e^{i(k_2 z - \omega t)}\,\hat{z}\times\hat{n}_T$$

$$= \frac{1}{v_2}\vec{\tilde{E}}_{0T}\, e^{i(k_2 z - \omega t)}\,(\cos\theta_T\,\hat{y} - \sin\theta_T\,\hat{x})$$

applying B-Cs # (iii) ; ($\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$) at $z=0$, we get

$$\tilde{E}_{0I}\,\hat{x} + \tilde{E}_{0R}\,\hat{n}_R = \tilde{E}_{0T}\,\hat{n}_T$$

$$\Rightarrow \tilde{E}_{0I}\,\hat{x} + \tilde{E}_{0R}(\cos\theta_R\,\hat{x} + \sin\theta_R\,\hat{y}) = \tilde{E}_{0T}(\cos\theta_T\,\hat{x} + \sin\theta_T\,\hat{y})$$

$\Rightarrow$ equating coeffs of $\hat{x}$ on both sides we get

$$\boxed{\tilde{E}_{0I} + \tilde{E}_{0R}\cos\theta_R = \tilde{E}_{0T}\cos\theta_T \quad \text{--- (1)}}$$ x-components

and equating coeffs of $\hat{y}$, we get

--- (2) y-components

Now applying B.Cs (iv) ; $\left(\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}\right)$ at $z=0$

we get

$$\frac{1}{\mu_1 v_1}\tilde{E}_{0I}\hat{y} - \frac{1}{\mu_1 v_1}\tilde{E}_{0R}(\cos\theta_R\,\hat{y} - \sin\theta_R\,\hat{x}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}(\cos\theta_T\,\hat{y} - \sin\theta_T\,\hat{x})$$

$$\Rightarrow \boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{0R}\sin\theta_R = \frac{-1}{\mu_2 v_2}\tilde{E}_{0T}\sin\theta_T \quad \text{----} \quad (3)} \quad x\text{- components}$$

and $$\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{0I} - \frac{1}{\mu_1 v_1}\tilde{E}_{0R}\cos\theta_R = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}\cos\theta_T \quad \text{---} (4)} \quad y\text{- components}$$

Now comparing equations (2) and (3)

$$\tilde{E}_{0R}\sin\theta_R = \tilde{E}_{0T}\sin\theta_T \quad \text{---} (2) \qquad ; \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

$$\tilde{E}_{0R}\sin\theta_R = -\beta\,\tilde{E}_{0T}\sin\theta_T \quad \text{---} (3)$$

and see that these two equations can be simultaneously satisfied with non-zero $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ is only when $\theta_R = \theta_T = 0$

clarification

i.e $\hat{n}_R = \cos\theta_R\,\hat{x} + \sin\theta_R\,\hat{y} = \cos(0)\,\hat{x} + \sin(0)\,\hat{y} = \hat{x}$, and

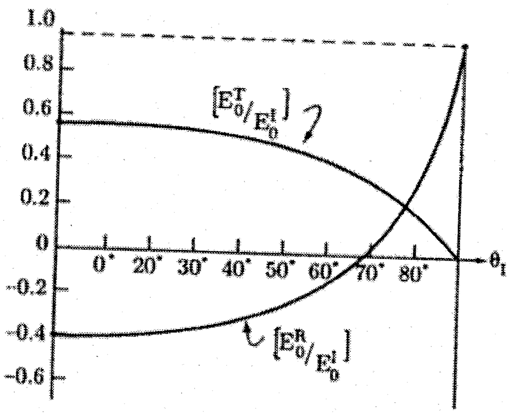
$\hat{n}_T = \cos\theta_T\,\hat{x} + \sin\theta_T\,\hat{y} = \cos(0)\,\hat{x} + \sin(0)\,\hat{y} = \hat{x}$, so from (2)

which says $\tilde{E}_{0R}\sin\theta_R = \tilde{E}_{0T}\sin\theta_T$, we know that $\tilde{E}_{0R} \neq \tilde{E}_{0T}$, so the only way that equation (2) is satisfied is when $\theta_R = \theta_T = 0$.

similar justification holds for equation (3).

Problem 9.18 The index of refraction of diamond is 2.42. Construct the graph analogous to Fig. 9.16 for the air/diamond interface. (Assume $\mu_1 = \mu_2 = \mu_0$.) In particular, calculate (a) the amplitudes at normal incidence, (b) Brewster's angle, and (c) the "crossover" angle, at which the reflected and transmitted amplitudes are equal.

$n_d = 2.42$, $n_{air} = 1$ $\Rightarrow \beta = \frac{\mu_1 n_2}{\mu_2 n_1}$



$\Rightarrow \beta \approx \frac{n_d}{n_{air}} = \frac{2.42}{1} = 2.42$

a) for normal incidence, $\theta_I = \theta_T = 0$

$\Rightarrow \alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{1}{1} = 1$; droping ~, we have

$$\Rightarrow E_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)E_{0I} = \left(\frac{1-2.42}{1+2.42}\right)E_{0I} = -0.42\,E_{0I}$$

and

$$E_{0T} = \left(\frac{2}{\alpha+\beta}\right)E_{0I} = \left(\frac{2}{1+2.42}\right)E_{0I} = 0.58\,E_{0I}$$

b) $\tan\theta_B = \frac{n_d}{n_{air}} \Rightarrow \theta = \tan^{-1}\left(\frac{2.42}{1}\right) = 67.55°$

c) cross-over angle occurs when $E_{0R} = E_{0T}$

$$\Rightarrow \left(\frac{\alpha-\beta}{\alpha+\beta}\right)E_{0I} = \left(\frac{2}{\alpha+\beta}\right)E_{0I} \Rightarrow \alpha-\beta = 2 \Rightarrow \alpha = 2+\beta = 2+2.42 = 4.42$$

now using $\alpha = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I} \Rightarrow \alpha\cos\theta_I = \sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}$

$\Downarrow$ { square both sides and use $\cos^2\theta_I = 1-\sin^2\theta_I$

$$\Rightarrow \alpha^2(1-\sin^2\theta_I) = 1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_I$$

$$\Rightarrow \sin^2\theta_I = \frac{1-\alpha^2}{\left(\frac{n_1}{n_2}\right)^2-\alpha^2} \Rightarrow \sin\theta_I = \sqrt{\frac{1-\alpha^2}{\left(\frac{n_{air}}{n_d}\right)^2-\alpha^2}} = \sqrt{0.969} = 0.984$$

$$\Rightarrow \theta_I = \sin^{-1}(0.984) \approx 79.9°$$

Problem 9.20 (a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $(2/\sigma)\sqrt{\epsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for (pure) water. (Use the static values of ϵ, μ, and σ; your answers will be valid, then, only at relatively low frequencies.) (b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength in the conductor). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7\ (\Omega.m)^{-1}$) in the visible range ($\omega \approx 10^{15}$/s), assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque? (c) Show that in a good conductor the magnetic field lags the electric field by 45°, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b)

a) $d = \frac{1}{\kappa}$ and $\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$, $\omega = 2\pi\nu$

using $(1+x)^{1/2} \approx 1 + \frac{1}{2}x$ $\Rightarrow \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1 + \frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2 - 1\right]^{1/2}$

$= \cancel{\omega}\frac{\sqrt{\varepsilon\mu}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\frac{\sigma}{\varepsilon\cancel{\omega}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$

$\Rightarrow d = \frac{1}{\kappa} = \frac{2}{\sigma}\sqrt{\varepsilon/\mu}$

for Pure water (DI): $\begin{cases} \varepsilon = \varepsilon_r\varepsilon_0 = 80.1\,\varepsilon_0 & \text{from table 4.2} \\ \mu = \mu_0(1+\chi_m) = \mu_0(1 - 9\times10^{-6}) \approx \mu_0 & \text{table 6.1} \\ \sigma = \frac{1}{\rho} = \frac{1}{8.3\times10^3} = 1.2\times10^{-4}(\Omega.m)^{-1} & \text{table 7.1} \end{cases}$

$\Rightarrow d = 2\times\underbrace{8.3\times10^3}_{\rho}\sqrt{\frac{80.1\times8.85\times10^{-12}}{4\pi\times10^{-7}}} = 394.3\text{ m}$

b) $\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\frac{\sigma}{\varepsilon\omega} - 1\right]^{1/2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} \Rightarrow d = \frac{1}{\kappa} = \sqrt{\frac{2}{\omega\mu\sigma}}$

$\Rightarrow d = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{10^{15}\times4\pi\times10^{-7}\times10^7}} \approx 1.26\times10^{-8}\text{ m}$

freq dependent

in addition, when $\left(\frac{\sigma}{\varepsilon\omega}\right)^2 \gg 1$, then from eq (9.126) $k \approx \kappa = \frac{2\pi}{\lambda}$

relations of k and κ

$\Rightarrow d = \frac{1}{\kappa} = \frac{\lambda}{2\pi}$

c) since $k \approx \kappa \Rightarrow \phi = \tan^{-1}\left(\frac{\kappa}{k}\right) = \tan^{-1}(1) = 45°$

and $B_0 = \frac{E_0}{\omega}K = \frac{E_0}{\omega}\sqrt{k^2+\kappa^2} \approx \frac{E_0}{\omega}\sqrt{2\kappa^2} = \frac{E_0\sqrt{2}}{\omega}\kappa$

$= \frac{E_0\sqrt{2}}{\omega}\sqrt{\frac{\omega\mu\sigma}{2}} = E_0\sqrt{\frac{\mu\sigma}{\omega}} \Rightarrow \frac{B_0}{E_0} = \sqrt{\frac{\mu\sigma}{\omega}}$

Problem 9.21

(a) Calculate the (time-averaged) energy density of an electromagnetic plane wave in a conducting medium (Eq. 9.138). Show that the magnetic contribution always dominates. [*Answer:* $(k^2/2\mu\omega^2)E_0^2e^{-2\kappa z}$]

(b) Show that the intensity is $(k/2\mu\omega)E_0^2e^{-2\kappa z}$.

a) the following relations will be used in this problem

$$\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{x}\ ;$$

$$\vec{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{y}\ ;$$

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}; \quad \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

$$K = \sqrt{k^2+\kappa^2} = \omega\left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}; \quad \tan\phi = \frac{\kappa}{k}\ ; \quad B_0 = \frac{E_0}{\omega}K$$

so
$$u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) = \frac{1}{2}e^{-2\kappa z}\left[\varepsilon E_0^2\cos^2(kz-\omega t+\delta_E) + \frac{1}{\mu}B_0^2\cos^2(kz-\omega t+\delta_E+\phi)\right]$$

now $\langle\cos^2\rangle = 1/2$

$$\Rightarrow \langle u\rangle = \frac{1}{2}e^{-2\kappa z}\left[\frac{\varepsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right]$$

$$= \frac{1}{4}e^{-2\kappa z}\left[\varepsilon E_0^2 + \frac{1}{\mu}\frac{E_0^2}{\omega^2}K^2\right] =$$

$$= \frac{1}{4}e^{-2\kappa z}\left\{\varepsilon E_0^2 + \frac{1}{\cancel{\mu}}E_0^2\varepsilon\cancel{\mu}\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right\}$$

$$= \frac{1}{4}e^{-2\kappa z}\varepsilon E_0^2\underbrace{\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}_{=\frac{2k^2}{\omega^2\varepsilon\mu}} = \frac{1}{4}e^{-2\kappa z}\varepsilon E_0^2\cdot\frac{2k^2}{\omega^2\varepsilon\mu}$$

$$= \frac{k^2}{2\mu\omega^2}E_0^2e^{-2\kappa z}$$

the ratio of $\langle u_m \rangle$ to $\langle u_e \rangle$ is

$$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \frac{B_0^2/\mu}{\varepsilon E_0^2} = \frac{1}{\varepsilon\mu E_0^2}\,\frac{E_0^2}{\omega^2}K^2 = \frac{1}{\varepsilon\mu E_0^2}\,\frac{E_0^2}{\omega^2}\,\omega^2\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

$$= \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} > 1$$

b) $I = \langle \vec{S} \rangle$; $\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$

$$\Rightarrow \vec{S} = \frac{1}{\mu}E_0 B_0 \cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\, e^{-2\kappa z}\,\hat{z}$$

$$\cos(kz-\omega t+\delta_E+\phi) = \cos(kz-\omega t+\delta_E)\cos\phi - \sin(kz-\omega t+\delta_E)\sin\phi$$

the second term will average to zero as $\langle \sin f(t)\cos f(t)\rangle = 0$ over one full period, so

$\underbrace{\sin f(t)\cos f(t)}_{\text{odd}}$

$$\vec{S} = \frac{1}{\mu}E_0 B_0 \cos^2(kz-\omega t+\delta_E)\cos\phi\, e^{-2\kappa z}\,\hat{z}$$

$$\Rightarrow I = \langle \vec{S} \rangle$$

$$= \frac{1}{\mu}E_0 B_0\left(\frac{1}{2}\right)\cos\phi\, e^{-2\kappa z}\,\hat{z}$$

now using $\cos\phi = \frac{k}{K}$ and $B_0 = \frac{E_0}{\omega}K$, we get

$$I = \langle \vec{S} \rangle = \frac{1}{\mu}E_0\frac{E_0}{\omega}K\left(\frac{1}{2}\right)\frac{k}{K}\, e^{-2\kappa z}\,\hat{z}$$

$$= \frac{k}{2\mu\omega}E_0^2\, e^{-2\kappa z}\,\hat{z}$$

Problem 9.22 Calculate the reflection coefficient for light at an air-to-silver interface ($\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon_0$, $\sigma = 6 \times 10^7 (\Omega \cdot \mathrm{m})^{-1}$), at optical frequencies ($\omega = 4 \times 10^{15}/\mathrm{s}$).

$$R = \left|\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^*, \text{ where}$$

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2 = \frac{\mu_1 v_1}{\mu_2 \omega}(k_2 + i\kappa_2) \text{ ; where}$$

$$k_2 = \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon_2\omega}\right)^2}+1\right]^{1/2} ; \quad \kappa_2 = \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon_2\omega}\right)^2}-1\right]^{1/2}$$

but silver is a good conductor ($\sigma \gg \epsilon_2\omega$), so

$$k_2 \approx \kappa_2 = \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\sqrt{\frac{\sigma}{\epsilon_2\omega}} = \sqrt{\frac{\mu_2\sigma\omega}{2}}$$

$$\Rightarrow \tilde{\beta} = \underbrace{\frac{\mu_1 v_1}{\mu_2\omega}\sqrt{\frac{\mu_2\sigma\omega}{2}}}(1+i) \quad ; \text{ but}$$

$$\frac{\mu_1 v_1}{\mu_2\omega}\sqrt{\frac{\mu_2\sigma\omega}{2}} = \mu_1 v_1\sqrt{\frac{\mu_2\sigma\omega}{2\mu_2^2\omega^2}} = \mu_1 v_1\sqrt{\frac{\sigma}{2\mu_2\omega}} = v_1\sqrt{\frac{\mu_1^2\sigma}{2\mu_2\omega}} ; \quad \mu_1 = \mu_2 \approx \mu_0$$

$$= c\sqrt{\frac{\mu_1\sigma}{2\omega}} = c\sqrt{\frac{\mu_0\sigma}{2\omega}} = 3\times10^8\sqrt{\frac{4\pi\times10^{-7}\times6\times10^7}{2\times4\times10^{15}}} = 29$$

$$\Rightarrow \tilde{\beta} = 29(1+i) = 29 + 29i$$

$$\Rightarrow R = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^* = \left(\frac{1-29-29i}{1+29+29i}\right)\left(\frac{1-29+29i}{1+29-29i}\right)$$

$$= \left(\frac{-28-29i}{30+29i}\right)\left(\frac{-28+29i}{30+29i}\right) = \frac{\sqrt{(-28)^2+(29)^2}}{\sqrt{30^2+29^2}}$$

$$= \frac{40.31}{41.72} = 0.97 \quad ;$$

97% of the light is reflected.

Problem 9.29 Consider a rectangular wave guide with dimensions 2.28 cm × 1.01 cm. What TE modes will propagate in this wave guide, if the driving frequency is 1.70×10^{10} Hz? Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use? What are the corresponding wavelengths (in open space)? $a = 2.28\,\text{cm}$, $b = 1.01\,\text{cm}$, $\omega_{mn} = 2\pi\nu_{mn}$

$$\Rightarrow \nu_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c\pi}{2\pi}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$\nu_{10} = \frac{1}{2\pi}\omega_{10} = \frac{c}{2a} = 0.66 \times 10^{10}$ Hz yes ✓

$\nu_{20} = \frac{1}{2\pi}\omega_{20} = 2\frac{c}{2a} = 1.32 \times 10^{10}$ Hz yes ✓

$\nu_{30} = \frac{1}{2\pi}\omega_{30} = 3\frac{c}{2a} = 1.97 \times 10^{10}$ Hz NO ×

$\nu_{01} = \frac{1}{2\pi}\omega_{01} = \frac{c}{2b} = 1.49 \times 10^{10}$ Hz yes ✓

$\nu_{02} = \frac{1}{2\pi}\omega_{02} = 2\frac{c}{2b} = 2.97 \times 10^{10}$ Hz NO ×

$\nu_{11} = \frac{1}{2\pi}\omega_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 1.62 \times 10^{10}$ Hz yes ✓

$\nu_{21} = \frac{1}{2\pi}\omega_{21} = \frac{c}{2}\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 1.98 \times 10^{10}$ Hz NO

$\nu_{30} = \frac{1}{2\pi}\omega_{30} = 3\frac{c}{2a} = 1.97 \times 10^{10}$ Hz NO

all the rest modes yield higher frequencies, so only 4 modes can propagate. (10, 20, 01, 11).

- to get only one mode, you must drive the wave guide at a frequency between ν_{10} and ν_{20}; i.e

$0.66 \times 10^{10} < \nu < 1.32 \times 10^{10}$ Hz. Now $\lambda = \frac{c}{\nu}$, so

$\lambda_{10} = \frac{c}{\nu_{10}} = \frac{c}{c/2a} = 2a$ and $\lambda_{20} = \frac{c}{\nu_{20}} = \frac{c}{c/a} = a$

$\Rightarrow \quad a < \lambda < 2a \quad \Rightarrow \quad 2.28 < \lambda < 4.56$ cm