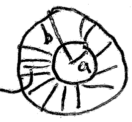
Electromagnetic theory (2)

HW # 15 - solution

Dr. Gassem Alzoubi

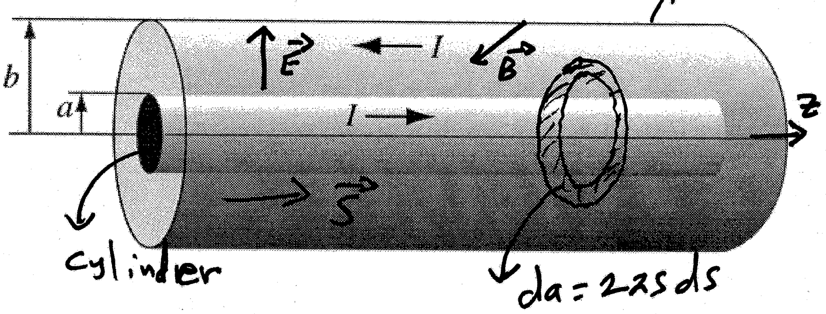
Power crossing $\perp$ area



Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V , and carry current I (down one and back up the other)

from Ex 7.13, we found

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \quad \text{and} \quad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$



$$\text{so } \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}\underbrace{(\hat{s}\times\hat{\phi})}_{\hat{k}}$$

$$= \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}\hat{k} \quad \text{, so}$$

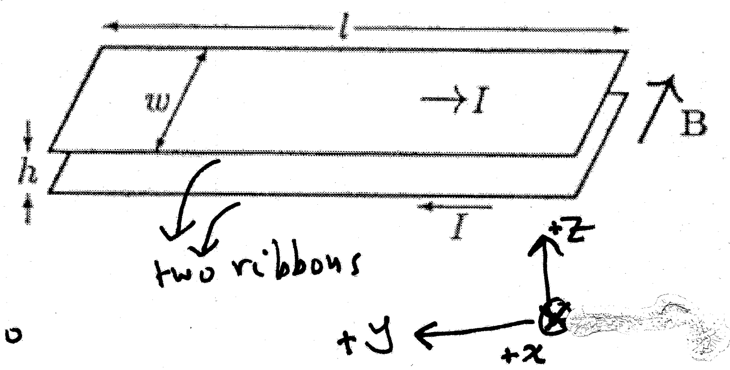
$$P = \int \vec{S}\cdot d\vec{a} = \int_a^b S\, 2\pi s\, ds = \frac{\lambda I}{2\pi\varepsilon_0}\int_a^b \frac{ds}{s} = \frac{\lambda I}{2\pi\varepsilon_0}\ln(b/a) \quad \text{--- (1)}$$

$$\text{But } \Delta V = \int_a^b \vec{E}\cdot d\vec{l} = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{ds}{s} = \frac{\lambda}{2\pi\varepsilon_0}\ln(b/a)$$; substitute this in (1)

we get $P = \frac{\lambda}{2\pi\varepsilon_0}\ln(b/a)\, I = \Delta V I$ as expected

b) $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{k}$

$\vec{B} = \mu_0 K\hat{i} = \frac{\mu_0 I}{w}\hat{i}$



$$\text{so } \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\sigma I}{\varepsilon_0 w}\hat{j} \quad \text{, so}$$

$$P = \int \vec{S}\cdot d\vec{a} = S\,(wh) = \frac{\sigma I}{\varepsilon_0 \cancel{w}}\cancel{w} h = \frac{\sigma I h}{\varepsilon_0} \quad \text{-- (1)}$$

but $\Delta V = \int \vec{E}\cdot d\vec{l} = Eh = \frac{\sigma}{\varepsilon_0}h$, so substitute in (1)

$$P = \frac{\sigma h}{\varepsilon_0} I = \Delta V I \quad \text{as expected.}$$

Problem: A long solenoid with length L, radius R and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\phi}$ direction, with $dI/dt = k$, where k is a constant. Using the results obtained in problem 7.15,

(a) Find the energy density u_{em} and the Poynting vector $\vec{S}$. Note especially the direction of $\vec{S}$. Check that Eq. 8.12 is satisfied.

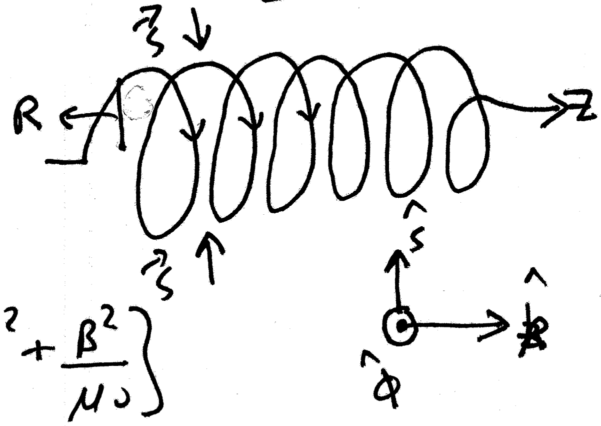
from problem 7.15, we found

$$\vec{B} = \begin{cases} \mu_0 n I \hat{k}, & s < R \\ 0, & s > R \end{cases} \quad \text{and} \quad \vec{E} = \begin{cases} -\frac{\mu_0 n s}{2} \frac{dI}{dt} \hat{\phi} = -\frac{\mu_0 n s k}{2} \hat{\phi}, & s < R \\ -\frac{\mu_0 n R^2}{2s} \frac{dI}{dt} \hat{\phi} = -\frac{\mu_0 n a^2}{2s} k \hat{\phi}, & s > R \end{cases}$$

$\vec{S} = 0$ outside, since $\vec{B} = 0$ outside, so inside the solenoid, we have

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(-\frac{\mu_0 n s k}{2} \cdot \mu_0 n I \, [\hat{\phi} \times \hat{k}]\right) = -\frac{\mu_0 n^2 I s}{2} k \hat{s}$$

$\vec{S}$ points to the central axis of the solenoid.



- the energy density u_{em} is

$$u_{em} = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} = \frac{1}{2}\left[\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right]$$

$$= \frac{1}{2}\left[\frac{\varepsilon_0 \mu_0^2 n^2 s^2}{2} k^2 + \frac{\mu_0^2 n^2 I^2}{\mu_0}\right]$$

Now eq^n 8.12 states that $\frac{\partial u_{em}}{\partial t} = -\nabla \cdot \vec{S}$, let us verify this eq^n

$$\frac{\partial u_{em}}{\partial t} = \frac{1}{2}\left[0 + \frac{\mu_0^2 n^2}{\mu_0} 2I \underbrace{\frac{dI}{dt}}_{k}\right] = \mu_0 n^2 I k, \quad \text{and}$$

$$-\nabla \cdot \vec{S} = -\frac{1}{s}\frac{\partial}{\partial s}\left(s \cdot \left[-\frac{\mu_0 n^2 I}{2} s \underbrace{\frac{dI}{dt}}_{k}\right]\right) = \frac{\mu_0 n^2 I k}{2s} \frac{\partial}{\partial s}(s^2)$$

$$= \mu_0 n^2 I k \checkmark$$

b) Find the total energy in the solenoid W_{em}

$$W_{em} = \int_V u_{em}\, d\tau = \int u_{em}\, s\,ds\,d\phi\,dz = \int_0^{2\pi} d\phi \int_0^{l} dz \int u_{em}\, s\,ds$$

$$= 2\pi l \int u_{em}\, s\,ds = \frac{2\pi l}{2}\int_0^R \left[\frac{\epsilon_0\mu_0^2 n^2 k^2 s^2}{2} + \frac{\mu_0^2 n^2 I^2}{\mu_0}\right] s\,ds$$

$$= \pi l \left[\frac{\epsilon_0 \mu_0^2 n^2 k^2}{2}\int_0^R s^3\, ds + \mu_0 n^2 I^2 \int_0^R s\,ds\right]$$

$$= \pi l\left[\frac{\epsilon_0\mu_0^2 n^2 k^2}{2}\frac{R^4}{4} + \mu_0 n^2 I^2 \frac{R^2}{2}\right]$$

$$= \frac{\pi l R^2 \mu_0 n^2}{2}\left[\frac{\epsilon_0\mu_0 R^2}{4}k^2 + I^2\right];$$

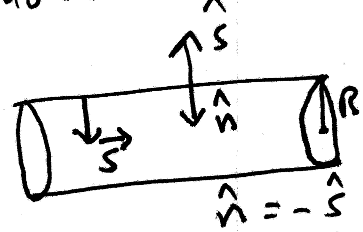
Now eq^n 8.9 reads $\frac{dW}{dt} = -\frac{dW_{em}}{dt} - \oint_S \vec{S}\cdot d\vec{a}$

inside the solenoid, $W = 0$ as there is no charge, so

⇒ let us verify eq^n 8.9 $\frac{dW_{em}}{dt} = -\oint_{s=R}\vec{S}\cdot d\vec{a}$

$$\frac{dW_{em}}{dt} = \frac{\pi l R^2\mu_0 n^2}{2}\left[0 + 2I \underbrace{\frac{dI}{dt}}_{k}\right] = \mu_0\pi l R^2 n^2 I k, \text{ and}$$

$-\oint \vec{S}\cdot d\vec{a} = -\oint \vec{S}\cdot\hat{n}\, da$ over cylindrical surface of radius R



$$= -\, S\Big|_{s=R} A$$

$$= -\, -\frac{\mu_0 n^2 I R k}{2}\cdot 2\pi R l$$

$= \mu_0 \pi l R^2 n^2 I R$ ✓, so $\frac{dW_{em}}{dt} = -\oint \vec{S}\cdot d\vec{a} = P_{in}$,

Note that $\vec{S}$ and $\hat{n}$ point in the same direction $\vec{S}\cdot\hat{n} = S$

= rate of increase of electromagnetic energy in