Electromagnetic theory (2)

HW # 14 – Solution

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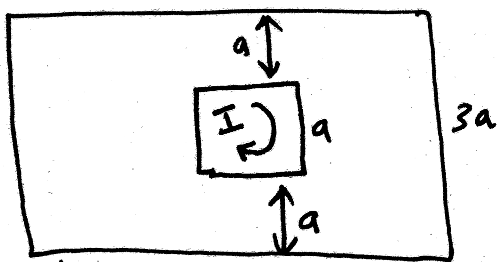
Problem 7.23 A square loop of wire, of side a, lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k$ (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

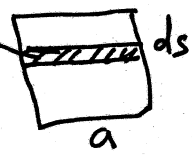
$$\mathcal{E} = -\frac{d\phi}{dt} = -M\frac{dI}{dt} = -Mk$$



but it is hard to calculate M using a current in the little loop, so using $M_{21} = M_{12}$, we can find the flux through the little loop when the same current I flows in the big loop $\Rightarrow$ Now the field of one long wire is $B = \frac{\mu_0 I}{2\pi s}$ and its flux through the little loop is $\phi_1 =$

$$\phi_1 = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 I}{2\pi}\int_a^{2a} \frac{1}{s}\, a\, ds$$

$da = a\,ds$



$$= \frac{\mu_0 I a}{2\pi}\ln 2 \Rightarrow \text{total flux } \phi = 2\phi_1 = \frac{\mu_0 I a}{\pi}\ln 2$$

$$\Rightarrow \phi = \frac{\mu_0 a \ln 2}{\pi} I = MI \Rightarrow M = \frac{\mu_0 a \ln 2}{\pi} \Rightarrow \mathcal{E} = -\frac{\mu_0 a k \ln 2}{\pi}$$

The net flux (through the big loop), due to I in the little loop is into the page. So the net flux is into the page.

This flux is increasing as $\phi \propto I$ (I increases uniformly),

so the induced current in the big loop will be in a direction to oppose this increase in flux. This requires that the induced field created by the big loop is out of page. Therefore, the induced current must flow counter clockwise (opposite to the current in little loop)

Problem 7.28 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.30 (you found L in Prob. 7.24); (b) using Eq. 7.31 (we worked out **A** in Ex. 5.12); (c) using Eq. 7.35; (d) using Eq. 7.34 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).

a) $W = \frac{1}{2} L I^2$; need L from problem 7.24, we have $B = \mu_0 n I$ inside the solonoid, so the flux through a single turn is $\Phi_1 = BA = \mu_0 n I \pi R^2$. in a length l, there are nl such turns, so the total flux is

$$\Phi = nl\,\Phi_1 = \mu_0 n^2 \pi R^2 l\, I \equiv LI \quad \Rightarrow L = \mu_0 n^2 \pi R^2 l$$

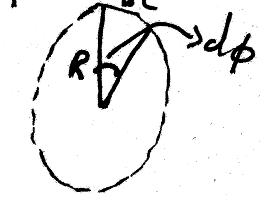
$$\Rightarrow W = \tfrac{1}{2} \mu_0 n^2 \pi R^2 l I^2$$

b) $W = \frac{1}{2} \oint \underbrace{(\vec{A} \cdot \vec{I})}\, dl$; $\vec{A} = \begin{cases} \frac{\mu_0 n I}{2} s\, \hat{\phi} & ; s < R \\ \frac{\mu_0 n I}{2} \frac{R^2}{s} \hat{\phi} & ; s > R \end{cases}$

the integrand is zero everywhere except at the surface ($s = R$).
this expression gives the energy of a single turn, but there are nl turns, so

$$W = \frac{nl}{2} \oint \vec{A} \cdot \vec{I}\, dl$$; it should be calculated over a circular loop on the surface

$$= \frac{nl}{2} \int_0^{2\pi} \frac{\mu_0 n I}{2} R\, I\, (R\, d\phi)$$; $dl = R\, d\phi$; $\vec{I} = I \hat{\phi}$



$$= \frac{1}{2} \mu_0 n^2 \pi R^2 l I^2$$

c) $W = \frac{1}{2\mu_0} \int B^2 d\tau$; $\vec{B} = \begin{cases} \mu_0 n I\, \hat{k} & ; s < R \\ 0 & ; s > R \end{cases}$

$$= \frac{B^2}{2\mu_0} \int d\tau = \frac{B^2}{2\mu_0} \pi R^2 l = \frac{1}{2} \mu_0 n^2 \pi R^2 l I^2$$

d) $W = \frac{1}{2\mu_0}\left[\int_V B^2\, d\tau - \oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}\right]$

here integrate over a cylindrical shell of inner radius $(a<R)$ and outer radius $(b>R)$

the first volume integral is $I = \int_V B^2\, d\tau$

$a<s<R$
$0<\phi<2\pi$
$0<z<l$

$$I = \int (\mu_0 n I)^2\, s\,ds\,d\phi\,dz$$

$$= (\mu_0 n I)^2 \int_0^{2\pi} d\phi \int_0^{l} dz \int_a^R s\,ds$$

we stop at $s=R$ as B_{out} = zero

$$= \pi \mu_0^2 n^2 I^2 l\,(R^2-a^2)$$

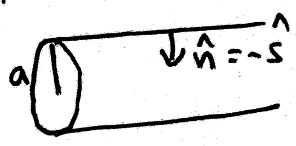
Now the second surface integral $II = \oint (\vec{A}\times\vec{B})\cdot d\vec{a}$
this integral must be evaluated at the inner surface $s=a$ as again at the outside surface of the tube $(s=b)$, the field is zero, so $\vec{A}\times\vec{B} = \frac{\mu_0 n I}{2} a\,\hat{\phi} \times \mu_0 n I\,\hat{k} = \frac{1}{2}\mu_0^2 n^2 I^2 a\,\hat{s}$

as $\hat{\phi}\times\hat{k} = \hat{s}$

so

$$II = \oint (\vec{A}\times\vec{B})\cdot d\vec{a} = -\int \frac{1}{2}\mu_0^2 n^2 I^2 a\; a\,d\phi\,dz$$

and $d\vec{a} = a\,d\phi\,dz\,(-\hat{s})$



$$= -\frac{1}{2}\mu_0^2 n^2 I^2 a^2 \int_0^{2\pi} d\phi \int_0^{l} dz$$

$$= -\frac{1}{2}\mu_0^2 n^2 I^2 a^2\, 2\pi l = -\mu_0^2 n^2 I^2 a^2 \pi l \text{, so}$$

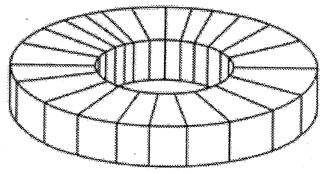
$$\Rightarrow W = \frac{1}{2\mu_0}\left[\pi\mu_0^2 n^2 I^2 l\,(R^2-a^2) + \mu_0^2 n^2 I^2 a^2 \pi l\right]$$

$$= \frac{1}{2}\mu_0 n^2 \pi R^2 l\, I^2$$

as expected

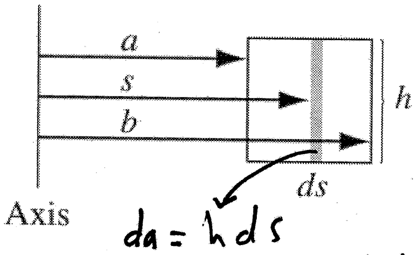
Problem 7.29: Calculate the energy stored in the toroidal coil of Ex. 7.11, by apply ing Eq. 7.35. Use the answer to check Eq. 7.28.

from example 7.11, $\vec{B} = \begin{cases} \frac{\mu_0 N I}{2\pi s}\hat{\phi}, & \text{inside coil} \\ 0, & \text{outside} \end{cases}$



$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau \; ; \quad d\tau = s\,ds\,d\phi\,dz \,, \text{ where}$$

$$= \frac{1}{2\mu_0}\int \frac{\mu_0^2 N^2 I^2}{4\pi^2 s^2}\, s\,ds\,d\phi\,dz \qquad \begin{array}{l} a < s < b \\ 0 < \phi < 2\pi \\ 0 < z < h \end{array}$$



$da = h\,ds$

$d\tau = (h\,ds)(s\,d\phi)$

$= h\,s\,ds\,d\phi$

$$= \frac{\mu_0 N^2 I^2}{8\pi^2}\int_a^b \frac{ds}{s}\int_0^{2\pi} d\phi \int_0^h dz$$

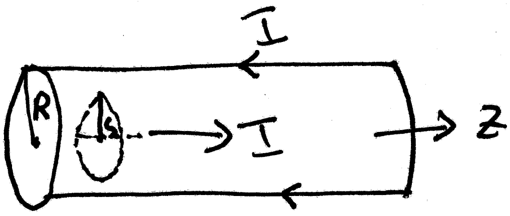
$$= \frac{1}{4\pi}\mu_0 N^2 h \ln(b/a)\, I^2 \equiv \tfrac{1}{2} L I^2$$

$$\Rightarrow L = \frac{1}{2\pi}\mu_0 N^2 h \ln(b/a)$$

Problem 7.30: A long cable carries current in one direction uniformly distributed over its (circular) cross section. The current returns along the surface (there is a very thin insulating sheath separating the currents). Find the self-inductance per unit length.

outside $B_{out} = 0$ as I_{enc} = zero

inside: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int J\, da$



$$B(2\pi s) = \mu_0 \frac{I}{\pi R^2}\pi s^2$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I s}{2\pi R^2}\hat{\phi} \; ; \text{ so } W = \frac{1}{2\mu_0}\int B^2 d\tau \; ; \quad d\tau = s\,ds\,d\phi\,dz$$

for cylindrical shell of radius s and length l and thickness ds

$$\Rightarrow W = \frac{1}{2\mu_0}\frac{\mu_0^2 I^2}{4\pi^2 R^2}\underbrace{\int_0^R s^3 ds}_{R^4/4}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\underbrace{\int_0^l dz}_{l}$$

$$= \frac{\mu_0 l}{16\pi} I^2$$

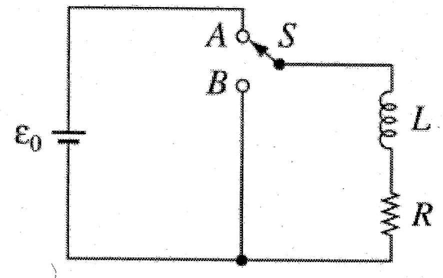
$$\equiv \tfrac{1}{2} L I^2 \quad \Rightarrow L = \frac{\mu_0 l}{8\pi} \quad \Rightarrow \frac{L}{l} = \frac{\mu_0}{8\pi} = \text{constant}$$

Problem 7.31 Suppose the circuit in figure has been connected for a long time when suddenly, at time t = 0, switch is thrown from A to B, bypassing the battery.

(a) What is the current at any subsequent time t?

(b) What is the total energy delivered to the resistor?

(c) Show that this is equal to the energy originally stored in the inductor.



initially; we have $\mathcal{E}_0 = V_L + V_R$

$\Rightarrow \mathcal{E}_0 = L\frac{dI}{dt} + IR$ --- (1) ; Now moving the switch from A→B means removing the battery from the circuit, so the magnetic energy stored in the inductor starts discharging through the resistor, so

$0 = L\frac{dI}{dt} + IR \Rightarrow -L\frac{dI}{dt} = IR \Rightarrow \frac{dI}{I} = -\frac{R}{L}dt$; integrating

$\int_{I_0}^{I}\frac{dI}{I} = -\frac{R}{L}\int_0^t dt \Rightarrow I(t) = I_0 e^{-\frac{R}{L}t}$; now at t=0 (A–S) state

the circuit was connected for a long time, so the current is steady (constant) and given by $\mathcal{E}_0 = \cancel{V_L} + V_R = V_R = I_0 R$ as

$V_L = L\underbrace{\frac{dI}{dt}}_{\text{zero}} = 0$ $\Rightarrow \mathcal{E}_0 = I_0 R \Rightarrow I_0 = \frac{\mathcal{E}_0}{R}$

$\Rightarrow I(t) = I_0 e^{-\frac{R}{L}t}$

$= \frac{\mathcal{E}_0}{R} e^{-\frac{t}{\tau}}$; $\tau = \frac{L}{R}$

b) the total energy delivered to the resistor is the same as the power disipated in the resistor over all time

$W_{res} = \int_0^\infty I^2 R\,dt = \int_0^\infty \frac{\mathcal{E}_0^2}{R^2} e^{-\frac{2t}{\tau}} dt = \frac{1}{2}\frac{\mathcal{E}_0^2}{R^2}L$

c) $W_L = \frac{1}{2} L I_0^2 = \frac{1}{2} L \frac{\mathcal{E}_0^2}{R^2}$; which matches part (b)

Problem 7.36 Refer to Prob. 7.16, to which the correct answer was

$$\mathbf{E}(s,t) = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right) \hat{\mathbf{z}}.$$

(a) Find the displacement current density $\mathbf{J}_d$.

(b) Integrate it to get the total displacement current,

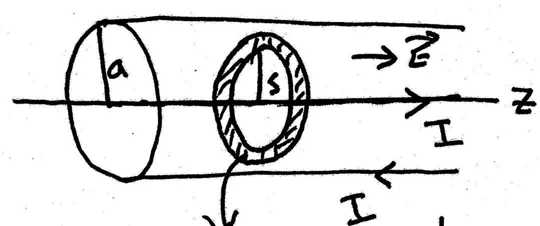
$$I_d = \int \mathbf{J}_d \cdot d\mathbf{a}.$$

(c) Compare I_d and I. (What's their ratio?) If the outer cylinder were, say, 2 mm in diameter, how high would the frequency have to be, for I_d to be 1% of I? [This problem is designed to indicate why Faraday never discovered displacement currents, and why it is ordinarily safe to ignore them unless the frequency is extremely high.]

a) $\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\varepsilon_0 \mu_0 I_0 \omega^2}{2\pi} \cos(\omega t) \ln\left(\frac{a}{s}\right) \hat{k}$

but from problem 7.16; $I = I_0 \cos\omega t \Rightarrow \vec{J}_d = \frac{\varepsilon_0\mu_0}{2\pi} \omega^2 I \ln\left(\frac{a}{s}\right) \hat{k}$

b) $I_d = \int \vec{J}_d \cdot d\vec{a} = \frac{\varepsilon_0\mu_0}{2\pi}\omega^2 I \int_0^a \ln\left(\frac{a}{s}\right) 2\pi s\, ds$



$da = 2\pi s\, ds$

$= \varepsilon_0\mu_0\omega^2 I \int_0^a (s\ln a - s\ln s)\, ds$

$= \varepsilon_0\mu_0\omega^2 I \left[(\ln a)\frac{s^2}{2} - \frac{s^2}{2}\ln s + \frac{s^2}{4} \right]_0^a$; using integral calculator

$= \frac{\varepsilon_0\mu_0 \omega^2 I a^2}{4}$

c) $\frac{I_d}{I} = \frac{\varepsilon_0\mu_0\omega^2 a^2}{4}$; but $\varepsilon_0\mu_0 = \frac{1}{c^2} \Rightarrow \frac{I_d}{I} = \left(\frac{\omega a}{2c}\right)^2$

now if $a = 10^{-3}$ m and $\frac{I_d}{I} = \frac{1}{100} \Rightarrow \frac{\omega a}{2c} = \frac{1}{10} \Rightarrow \omega = \frac{2c}{10a} = 6\times10^{10}/s$

$\Rightarrow \nu = \frac{\omega}{2\pi} = \frac{6\times10^{10}}{2\pi} \approx 10^{10}$ Hz or 10^4 MHz this is the microwave region which is well above radio frequencies region