Electromagnetic theory (2)

HW # 13 - Solution

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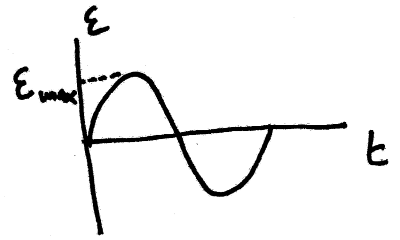
Problem 7.12: A long solenoid, of radius a, is driven by an alternating current, so that the field inside is sinusoidal: $\vec{B}(t) = B_0 \cos(\omega t)\hat{k}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

we see that $\vec{B}$ does not depend on coordinates, so $\phi = \int \vec{B}\cdot d\vec{a}$ gives

$$\Phi = \vec{B}\cdot\vec{A} = BA = B_0 \cos(\omega t)\, \pi (a/2)^2 = \frac{\pi a^2}{4} B_0 \cos(\omega t)$$

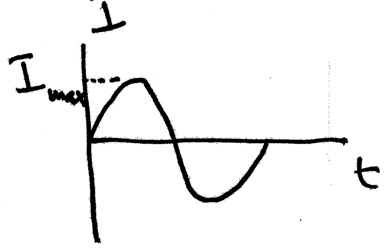
$$\Rightarrow \mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi a^2 B_0 \omega}{4} \sin \omega t = \mathcal{E}_{max} \sin(\omega t) \text{ ; where } \mathcal{E}_{max} = \frac{\pi a^2 B_0 \omega}{4}$$



and $$I(t) = \frac{\mathcal{E}}{R} = \frac{\pi a^2 \omega B_0}{4R} \sin(\omega t)$$

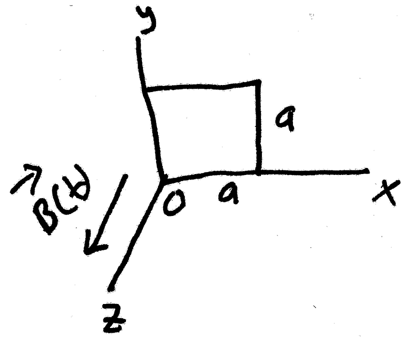
$$= I_{max} \sin(\omega t)$$

where $$I_{max} = \frac{\pi a^2 \omega B_0}{4R}$$



Problem 7.13: A square loop of wire, with sides of length a, lies in the first quadrant of the xy plane, with one corner at the origin. In this region, there is a nonuniform time-dependent magnetic field $\vec{B}(y, t) = ky^3t^2\hat{k}$ (where k is a constant). Find the emf induced in the loop.

we see that $\vec{B}$ depends on coordinates, so we have to use integration to find Φ



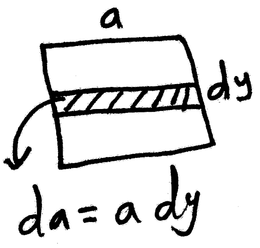
$$\Phi = \int \vec{B}\cdot d\vec{a} = \int B\,da = \int B\,dx\,dy$$

$$= kt^2 \int y^3 dx\,dy = kt^2 \int_0^a dx \int_0^a y^3 dy$$

$$= \frac{1}{4} k t^2 a^5, \text{ so}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{1}{2} k t a^5$$

another way to find Φ



$$\Phi = \int \vec{B}\cdot d\vec{a} = \int B\,da = \int_0^a B\,a\,dy$$

$$= kt^2 a \int_0^a y^3 dy$$

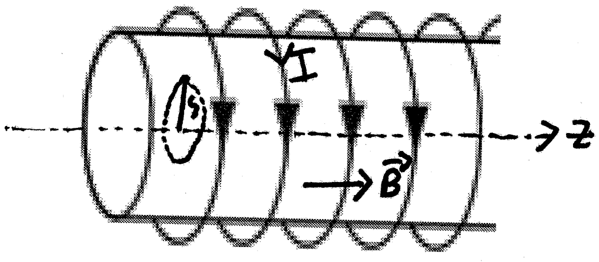
$$= \frac{1}{4} k t^2 a^5 \text{ as expected.}$$

Problem 7.15: A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the ϕ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

for solenoid, we have $\vec{B} = \begin{cases} \mu_0 n I \hat{k}, & s<a \text{ inside} \\ 0, & s>a \text{ outside} \end{cases}$

<u>inside:</u> for a circular amperian loop of radius s inside, we have

$$\Phi = \int \vec{B}\cdot d\vec{a} = \vec{B}\cdot\vec{A} = BA = B\pi s^2$$



$$= \mu_0 n I \pi s^2.$$

Now from Faraday's law

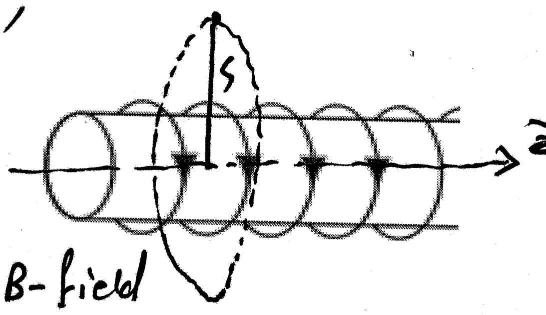
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} \Rightarrow (2\pi s)E = -\mu_0 n \pi s^2 \frac{dI}{dt}$$

$$\Rightarrow \vec{E} = -\frac{\mu_0 n s}{2}\frac{dI}{dt}\hat{\phi}$$; see that $-\frac{\partial \vec{B}}{\partial t}$ points in the $-z$-direction $(-\hat{k})$ [thumb direction], so the rest 4 fingers points in the $-\hat{\phi}$ direction

<u>outside:</u>

for an amperian loop of radius $s>a$, we have

$$\Phi = \vec{B}\cdot\vec{A} = BA = B\underbrace{\pi a^2} = \mu_0 n I \pi a^2$$



we take only area penetrated by the B-field

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

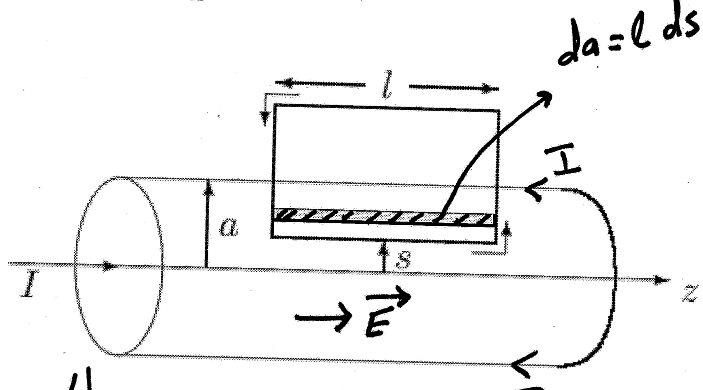
$$E(2\pi s) = -\mu_0 n \pi a^2 \frac{dI}{dt}$$

$$\Rightarrow \vec{E} = -\frac{\mu_0 n a^2}{2s}\frac{dI}{dt}\hat{\phi}$$

Problem 7.16: An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a. (a) In what direction does the induced electric field point (radial, circumferential, or longitudinal)? (b) Assuming that the field goes to zero as $s \to \infty$, find $\mathbf{E}(s, t)$

<u>inside:</u> the B field is due to I in the wire

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \frac{\mu_0 I_0}{2\pi s}\cos\omega t\ \hat{\phi}$$



$$\Rightarrow -\frac{\partial \vec{B}}{\partial t} = +\frac{\mu_0 I_0 \omega}{2\pi s}\sin(\omega t)\ \hat{\phi}$$

making thumb points in the $\hat{\phi}$ direction, then the curl of the 4 fingers (direction of $\vec{E}$) points in the + z-direction, so $\vec{E}$ is longitudinal. Now using Faraday's law

$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$, so using the amperian loop shown in figure,

we have $\Phi = \int \vec{B}\cdot d\vec{a} = \int_s^a \frac{\mu_0 I}{2\pi s} l\,ds = \frac{\mu_0 I l}{2\pi}\ln\left(\frac{a}{s}\right)$

$$= \frac{\mu_0 I_0 l}{2\pi}\ln\left(\frac{a}{s}\right)\cos\omega t$$

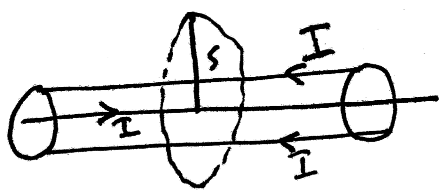
$$\Rightarrow E_{in} l - \underbrace{\cancel{E_{out}}}_{0} l = -\frac{d\Phi}{dt}$$

↳ will be shown next

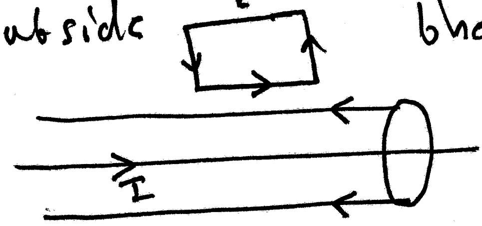
$$E_{in}\, l = -\frac{\mu_0 I_0 l}{2\pi}\ln\left(\frac{a}{s}\right)\omega(-\sin\omega t)$$

$$\Rightarrow \vec{E}_{in} = \frac{\mu_0 I_0 \omega}{2\pi}\ln(a/s)\sin(\omega t)\ \hat{k}$$

<u>outside:</u> taking a circular amperian loop outside gives $\vec{B}_{out} = 0$ as the net current inside is zero.



Now consider a rectangular amperian loop outside



then $\oint_C \vec{E}_{out}\cdot d\vec{l} = \frac{-d\Phi}{dt}$

but $\Phi = 0$ outside as $\vec{B}_{out} = \text{zero}$

$$\Rightarrow \oint \vec{E}_{out}\cdot d\vec{l} = 0$$

$$\Rightarrow \vec{E}_{out} = \text{zero}$$ as $d\vec{l}$ is arbitrary and $\neq 0$

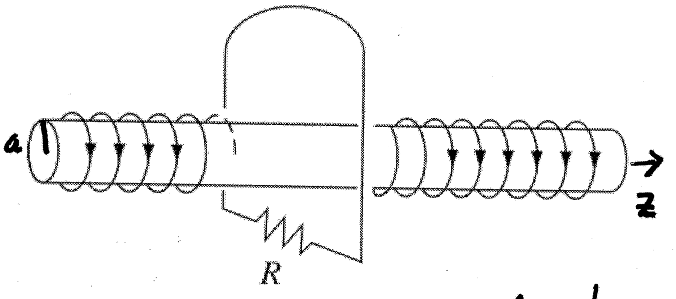
Problem 7.17: A long solenoid of radius a, carrying n turns per unit length, is looped by a wire with resistance R, as shown in figure.

(a) If the current in the solenoid is increasing at a constant rate (dI/dt = k), what current flows in the loop, and which way (left or right) does it pass through the resistor?

(b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop (toward the left, to a place far from the loop), what total charge passes through the resistor?

$\vec{B} = B\hat{k} = \mu_0 n I \hat{k}$

(a) $I = \frac{\mathcal{E}}{R}$; Need to find $\mathcal{E}$ first



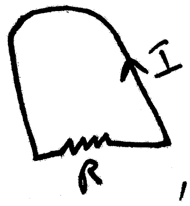
$\mathcal{E} = -\frac{d\Phi}{dt}$; $\Phi = \vec{B}\cdot\vec{A} = BA$

$= \mu_0 n I \pi a^2$

$\mathcal{E} = -\mu_0 n \pi a^2 \frac{dI}{dt} = -\mu_0 n \pi a^2 k$

the (−) sign indicates that the induced current in the loop is opposite in direction to the current in the soleniod, so

$I = \frac{\mathcal{E}}{R} = -\frac{\mu_0 n \pi a^2 k}{R}$; counter clock wise



Note that the induced I produces an induced magnetic field pointing to left, according to Lenz' law

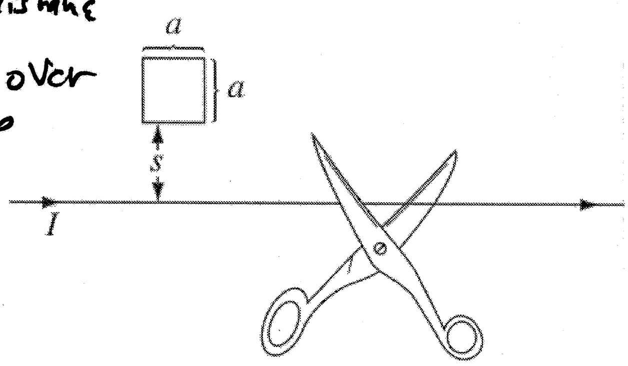
(b) $I = \frac{\mathcal{E}}{R} = \frac{1}{R}\left(-\frac{d\Phi}{dt}\right) \Rightarrow I(t)\,dt = -\frac{1}{R}\,d\Phi$, integrate

$\int_0^t I(t)\,dt = -\frac{1}{R}\int_{\Phi_i}^{\Phi_f} d\Phi \Rightarrow \Delta Q = -\frac{1}{R}\left[\Phi_f - \Phi_i\right]$

$\Phi_f \rightarrow$ zero

$\Rightarrow \Delta Q = \frac{1}{R}\Phi_i = \frac{\mu_0 n I \pi a^2}{R}$

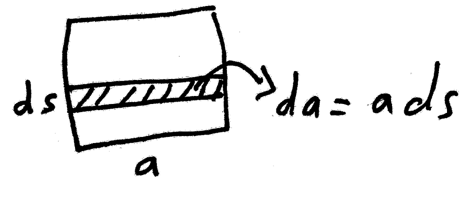
Problem 7.18: A square loop, side *a*, resistance R, lies a distance s from an infinite straight wire that carries current **I** (Fig. 7.29). Now someone cuts the wire, so **I** drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows?



before cutting the wire, the field at a distance s is $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$; non-uniform over the square loop

$$\Rightarrow \Phi = \int \vec{B}\cdot d\vec{a} = \int B\,da$$

$$= \int_s^{s+a} \frac{\mu_0 I}{2\pi s}\,a\,ds \ ;$$



$$= \frac{\mu_0 I a}{2\pi}\int_s^{s+a}\frac{ds}{s}$$

$$= \frac{\mu_0 I a}{2\pi}\ln\left(\frac{s+a}{s}\right) = \frac{\mu_0 I a}{2\pi}\ln(1+a/s) \ ;$$

now $I = \frac{\mathcal{E}}{R} = \frac{1}{R}\left(-\frac{d\Phi}{dt}\right) \Rightarrow I(t)\,dt = -\frac{1}{R}\,d\Phi$, integrate

$$\int_0^t I(t)\,dt = -\frac{1}{R}\int d\Phi \Rightarrow \Delta Q = -\frac{1}{R}[\underbrace{\Phi_f}_{\text{zero}} - \Phi_i] = \frac{\Phi_i}{R}$$

$$= \frac{\mu_0 a I}{2\pi R}\ln(1+a/s)$$

if $s = a \Rightarrow \Delta Q = \frac{\mu_0 a I}{2\pi R}\ln 2$

once the wire is cut, the field through the loop starts decreasing and the flux does so. To compensate for the decrease in the flux through the loop, an induced current must flow counter clockwise to generate an induced B field out of page to compensate for the decrease in the flux.