Electromagnetic theory (2)

HW # 12 - Solution

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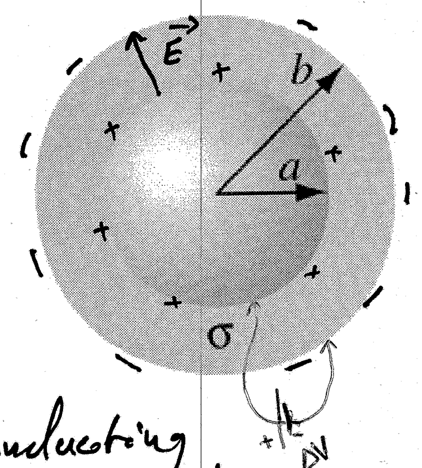
Problem 7.1 Two concentric metal spherical shells, of radius a and b, respectively, are separated by weakly conducting material of conductivity σ.

(a) If they are maintained at a potential difference ΔV, what current flows from one to the other?

(b) What is the resistance between the shells?

let us place a charge $+Q$ on the inner shell and a charge $-Q$ on the outer one. the field $\vec{E}$ is radially outward, so



$$\vec{E} = E_r \hat{r} = k_e \frac{Q}{r^2} \hat{r}$$

notice that the field between the two shells is not affected by the weakly conducting material in between as ΔV is maintained fixed, so

$$V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{r} = -k_e Q \int_b^a \frac{dr}{r^2} \; ; \; d\vec{r} = dr\,\hat{r}$$

$$= k_e Q \left(\frac{1}{a} - \frac{1}{b}\right)$$

so in order to maintaine this potential difference, we must place a $+Q$ on the inner shell given by $Q = \dfrac{V_a - V_b}{k_e\left(\frac{1}{a} - \frac{1}{b}\right)}$, so

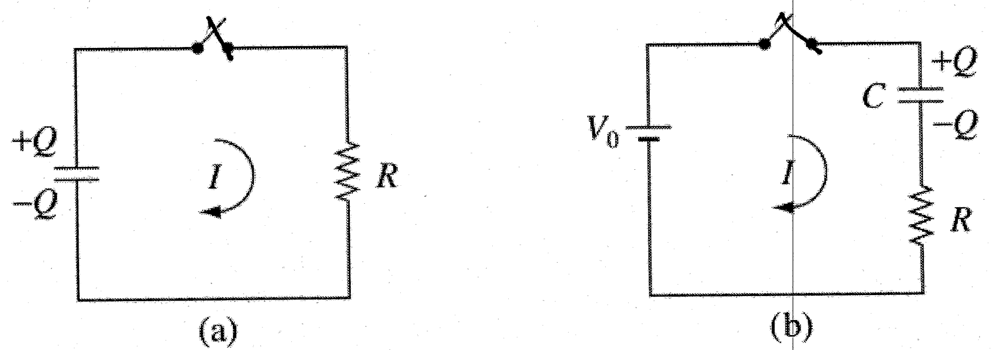
$I = \oint \vec{J} \cdot d\vec{a}$ over spherical surface of radius r

$$= \sigma \oint \vec{E} \cdot d\vec{a} = \sigma E \int da = \sigma E A = \sigma\, k_e \frac{Q}{r^2} \cdot 4\pi r^2$$

$$= \sigma \frac{1}{4\pi\varepsilon_0} Q\, 4\pi = \sigma \frac{Q}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \frac{V_a - V_b}{k_e\left(\frac{1}{a} - \frac{1}{b}\right)} = 4\pi\sigma \frac{(V_a - V_b)}{\frac{1}{a} - \frac{1}{b}}$$

b) $$R = \frac{\Delta V}{I} = \frac{V_a - V_b}{I} = \frac{(V_a - V_b)}{\dfrac{4\pi\sigma (V_a - V_b)}{\left(\frac{1}{a} - \frac{1}{b}\right)}} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Problem 7.2 A capacitor C has been charged up to potential V_0; at time t = 0, it is connected to a resistor R, and begins to discharge (Fig. 7.5a).



(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

(b) What was the original energy stored in the capacitor (Eq. 2.55)? By integrating Eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of voltage V_0, at time $t = 0$ (Fig. 7.5b).

(c) Again, determine $Q(t)$ and $I(t)$.

(d) Find the total energy output of the battery ($\int V_0 I\, dt$). Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? [Notice that the answer is independent of R!]

a) according to the loop rule in circuit (a), we have

$$\Delta V_C + \Delta V_R = 0 \Rightarrow \frac{Q}{C} + IR = 0 \Rightarrow I = -\frac{Q}{RC}$$

$$\Rightarrow \frac{dQ}{dt} = -\frac{Q}{RC} \Rightarrow \frac{dQ}{Q} = \frac{-dt}{RC} \Rightarrow \text{integrating}$$

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt \Rightarrow \ln Q\Big|_{Q_0}^{Q} = -\frac{t}{RC}$$

$$\Rightarrow \ln\frac{Q}{Q_0} = e^{-t/RC} \Rightarrow Q(t) = Q_0 e^{-t/RC}$$

but at $t=0$, $Q(t=0) = Q_0 = CV_0$

$$\Rightarrow Q(t) = CV_0 e^{-t/RC} \quad \ldots (1)$$

$$I = \frac{dQ}{dt} = -\frac{CV_0}{RC} e^{-t/RC} = -\frac{V_0}{R} e^{-t/RC}$$

the minus sign indicates that the discharging current is opposite to the direction of the charging current.

b) $W_0 = \frac{1}{2} C V_0^2$ original energy stored in the capacitor.

Now the energy delivered to the ~~capacitor~~ risistor is

$$\int_0^\infty P\,dt = \int_0^\infty I^2 R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC} = \frac{V_0^2}{R}\left[-\frac{RC}{2} e^{-2t/RC}\right]_0^\infty$$

$$= \frac{1}{2} C V_0^2$$

so energy lost by the capacitor is dissipated in the resistor as a heat

c) this is the charging circuit where $V_0 = \Delta V_C + \Delta V_R$

$$V_0 = \frac{Q}{C} + IR \Rightarrow IR = V_0 - \frac{Q}{C} \Rightarrow I = \frac{V_0}{R} - \frac{Q}{RC}$$

$$\Rightarrow I = \frac{V_0 C}{RC} - \frac{Q}{RC} = \frac{V_0 C - Q}{RC} = -\frac{Q - V_0 C}{RC} = \frac{dQ}{dt}$$

$$\Rightarrow \frac{dQ}{Q - V_0 C} = -\frac{dt}{RC} \text{ ; integrating}$$

$$\int_0^Q \frac{dQ}{Q - V_0 C} = -\frac{1}{RC}\int_0^t dt \Rightarrow \ln(Q - V_0 C)\Big|_0^Q = -\frac{t}{RC}$$

$$\Rightarrow \ln(Q - V_0 C) - \ln(-V_0 C) = -t/RC$$

$$\Rightarrow \ln \frac{Q - V_0 C}{-V_0 C} = -\frac{t}{RC} \Rightarrow \frac{Q - V_0 C}{-V_0 C} = e^{-t/RC}$$

$$\Rightarrow Q - V_0 C = -V_0 C e^{-t/RC} \Rightarrow \boxed{Q(t) = V_0 C (1 - e^{-t/RC})} \quad \text{--(2)}$$

$$= Q_{max}(1 - e^{-t/\tau})$$

with $\tau = RC$

$$\Rightarrow I = \frac{dq}{dt} = V_0 C\left(0 - \left(-\frac{1}{RC}\right)e^{-t/RC}\right) = \frac{V_0}{R} e^{-t/RC}$$

Notice that this charging current is opposite to the discharging one

$$\int_0^\infty e^{-\frac{t}{RC}} dt = \left.\frac{e^{-\frac{t}{RC}}}{-\frac{1}{RC}}\right|_0^\infty$$

d) the total energy output of the battery is

$$\int_0^\infty P\,dt = \int_0^\infty V_0 I\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}\,dt = \frac{V_0^2}{R} RC = CV_0^2$$

again the energy delivered to the resistor is

$$\int_0^\infty P\,dt = \int_0^\infty I^2 R = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC} = \tfrac{1}{2} CV_0^2$$

so half of the energy output of the battery goes to the resistor ($\tfrac{1}{2}CV_0^2$) and the other half goes to the ~~resistor~~ capacitor ($\tfrac{1}{2}CV_0^2$)

Problem 7.3

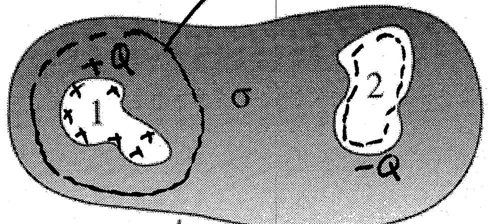
(a) Two metal objects are embedded in weakly conducting material of conductivity σ (Fig. 7.6). Show that the resistance between them is related to the capacitance of the arrangement by

$$R = \frac{\epsilon_0}{\sigma C}.$$

(b) Suppose you connected a battery between 1 and 2, and charged them up to a potential difference V_0. If you then disconnect the battery, the charge will gradually leak off. Show that $V(t) = V_0 e^{-t/\tau}$, and find the **time constant**, τ, in terms of ϵ_0 and σ.

a) let us assume conductor 1 carries a positive charge $+Q$ and conductor 2 carries a negative charge $-Q$, so

surface



$I = \oint \vec{J} \cdot d\vec{a}$; where the integral is taken over a surface enclosing $+Q$

using Gauss's law $\oint \vec{E} \cdot d\vec{A} = Q_{in}/\varepsilon_0$

$I = \oint \vec{J} \cdot d\vec{a} = \sigma \oint \vec{E} \cdot d\vec{a} = \frac{\sigma Q}{\varepsilon_0}$; but $Q = CV$ and $V = IR$ $\Rightarrow Q = CIR$

$\cancel{I} = \frac{\sigma}{\varepsilon_0} C \cancel{I} R \quad \Rightarrow R = \frac{\varepsilon_0}{\sigma C}$

b) during the charging, we have $V_0 = \frac{Q}{C} + IR$. after disconnecting the battery (set $V_0 = 0$)

$\Rightarrow 0 = \frac{Q}{C} + IR \Rightarrow 0 = \frac{Q}{C} + R\frac{dQ}{dt} \Rightarrow \frac{dQ}{dt} = -\frac{Q}{RC}$

$\Rightarrow Q(t) = Q_0 e^{-t/RC} = CV_0 e^{-t/RC} \Rightarrow V = \frac{Q}{C} = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$

$\therefore V(t) = V_0 e^{-t/RC}$

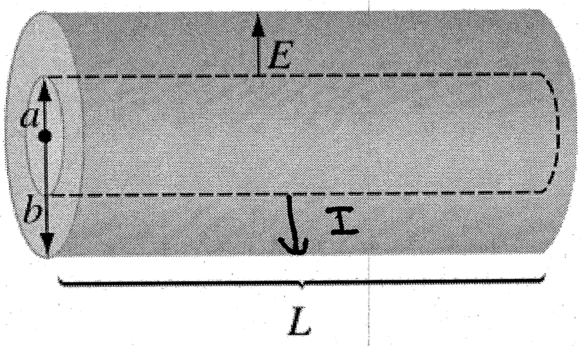
and $\tau = RC = \frac{\varepsilon_0}{\sigma \cancel{C}} \cancel{C} = \frac{\varepsilon_0}{\sigma}$

in this example, σ was constant, and hence we used Gauss's law

$$\oint \vec{E} \cdot d\vec{a} = q_{in}/\varepsilon_0$$

Problem 7.4 Suppose the conductivity of the material separating the cylinders in Ex. 7.2 is not uniform; specifically, $\sigma(s) = k/s$, for some constant k. Find the resistance between the cylinders. [*Hint:* Because σ is a function of position, Eq. 7.5 does not hold, the charge density is not zero in the resistive medium, and **E** does not go like $1/s$. But we *do* know that for steady currents I is the same across each cylindrical surface. Take it from there.]

the current flows from surface a to surface b in the radial direction, so



$J = \frac{I}{A}$; A: lateral area of a cylindrical shell of radius s

$$J = \frac{I}{2\pi s l} \Rightarrow E = \frac{J}{\sigma} = \frac{I}{2\pi \sigma s l} \text{ ; but } \sigma(s) = \frac{k}{s}$$

$$= \frac{I}{2\pi \frac{k}{s} s l} = \frac{I}{2\pi k l} \text{ ; } \vec{E} = E\hat{s}$$

$$\Delta V = V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{l} = -\int_b^a E_r\, dr = -\int_b^a E_s\, ds$$

$$= -\int_b^a \frac{I}{2\pi k l} ds = \frac{-I}{2\pi k l} s \Big|_b^a$$

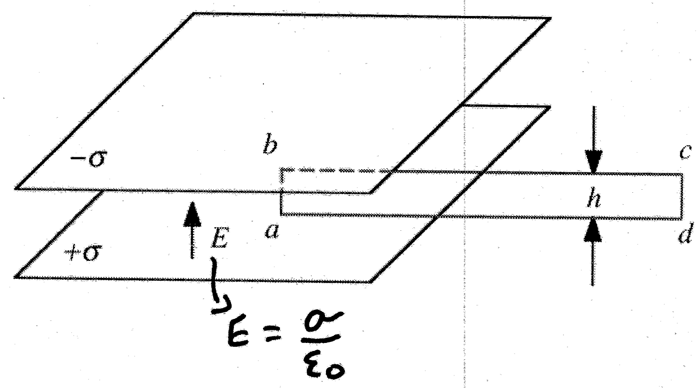
$$\Delta V = -\frac{I}{2\pi k l}[a-b] = \frac{I}{2\pi k l}[b-a] > 0$$

$$\Rightarrow R = \frac{\Delta V}{I} \doteq \frac{b-a}{2\pi k l}$$

Problem 7.5

a) Show that electrostatic force alone cannot be used to drive current around a circuit.

b) A rectangular loop of wire is situate so that one end is between the plates of a parallel-plate capacitor, oriented parallel to the field $E = \frac{\sigma}{\varepsilon_0}$. The other end is way outside, where the field is essentially zero. If the width of the loop is h and its total resistance is R, what current flows? Explain.



a) If only electrostatic forces are present then the force per unit charge is equal to the electrostatic force:

$$\vec{f} = \vec{E}$$

The associated emf is therefore equal to

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \vec{E} \cdot d\vec{l} = 0$$

b) The only force on the charge carriers in the wire loop is the electric force. However, in part a) we concluded that the emf associate with an electric force, generated by an electrostatic field, is equal to zero. Therefore, the emf in the wire loop is equal to zero, and consequently the current in the loop is also equal to zero. Note: at first sight it might appear that there is a net emf, if we assume that the electric field generated by the capacitor is that of an ideal capacitor (that is a homogeneous field inside and no field outside). Under that assumption, the emf is equal to

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} = Eh = \frac{\sigma}{\varepsilon_0} h$$

The contribution of the path integral from c to d is equal to zero since the electric field is zero there, and the contribution of the path integrals between b and c and between a and d is equal to zero since the electric field and the displacement are perpendicular there. Clearly the calculated emf is non-zero, and disagrees with the result of part a). The disagreement is a result of our incorrect assumption that the electric field outside the capacitor is equal to zero (there are fringing fields).

$\vec{E} \perp d\vec{l}$

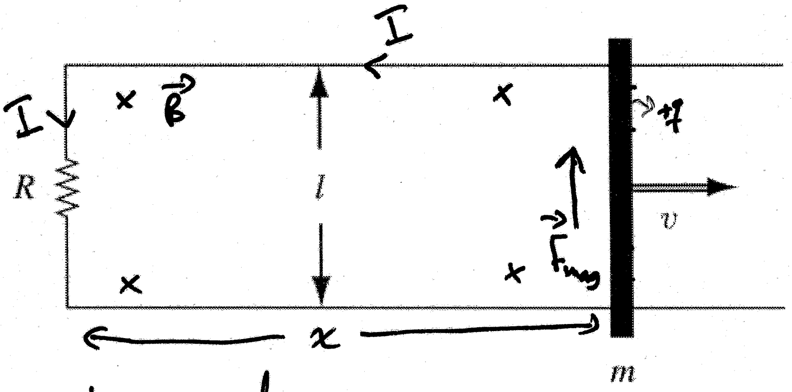
Problem 7.7 A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart . A resistor R is connected across the rails, and a uniform magnetic field **B**, pointing into the page, fills the entire region.

(a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

(d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.



a) $\phi = BA = Blx$

$$\mathcal{E} = -\frac{d\phi}{dt} = -Bl\frac{dx}{dt} = -Blv$$

the positive charges in the bar experience a magnetic force ($\vec{F}_{mag} = q\,\vec{v}\times\vec{B}$) directed upward, and hence driving a current counter clockwise

$$I = \frac{|\mathcal{E}|}{R} = \frac{Blv}{R}$$

b) the magnetic force acting on the bar is

$$\vec{F} = I\,\vec{l}\times\vec{B} = IlB \text{ to left}$$

$$= \frac{Blv}{R}lB = \frac{B^2l^2v}{R} \quad \text{or } \vec{F} = -\frac{B^2l^2v}{R}\hat{i}$$

c) $\vec{F} = m\frac{d\vec{v}}{dt}$

$$-\frac{B^2l^2v}{R} = m\frac{dv}{dt} \Rightarrow \frac{dv}{v} = -\frac{B^2l^2}{mR}dt$$

integrating $\displaystyle\int_{v_0}^{v}\frac{dv}{v} = -\frac{B^2l^2}{mR}\int_0^t dt \Rightarrow \ln\frac{v}{v_0} = -\frac{B^2l^2}{mR}t$

$$\Rightarrow \frac{v}{v_0} = e^{-\frac{B^2 l^2}{mR} t} \Rightarrow \boxed{v(t) = v_0\, e^{-\frac{B^2 l^2}{mR} t}} = v_0 e^{-\alpha t}$$

with $\alpha = \frac{B^2 l^2}{mR}$

d) $K_i = \frac{1}{2} m v_0^2$

Now $P = \frac{dw}{dt} \Rightarrow dw = P\,dt = I^2 R\, dt$

$$= \frac{B^2 l^2 v^2}{R^2} R\, dt$$

$$= \frac{B^2 l^2}{R} v^2\, dt \quad ; \quad v = v_0 e^{-\alpha t}, \quad \alpha = \frac{B^2 l^2}{mR}$$

$$= \frac{B^2 l^2}{R} v_0^2\, e^{-2\alpha t}\, dt$$

So the total energy delivered to the resistor is

$$W = \int dw = \int_0^\infty P\, dt = \frac{B^2 l^2}{R} v_0^2 \int_0^\infty e^{-2\alpha t}\, dt = m\alpha v_0^2 \underbrace{\int_0^\infty e^{-2\alpha t}\, dt}_{\frac{1}{2\alpha}}$$

$$= \alpha m v_0^2 \frac{1}{2\alpha} = \frac{1}{2} m v_0^2$$

by direct integration or using $\int_0^\infty x^n e^{-ax}\, dx = \frac{1}{a^{n+1}}$

with $a = 2\alpha$ and $n = 0$

Problem 7.8 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown in Fig. 7.18.

Notice that the field produced by the current points out of page over the loop area

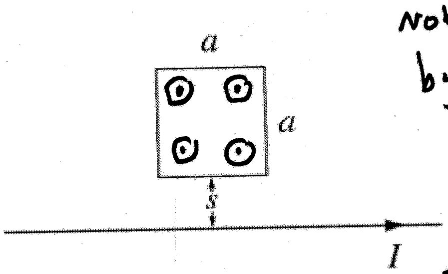


FIGURE 7.18

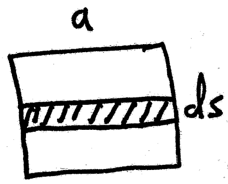
$$\vec{B}(s) = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

(a) Find the flux of **B** through the loop.

(b) If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

(c) What if the loop is pulled to the *right* at speed v?

a) Notice that the field is not uniform through the loop area as B depends on s, so



$da = a\,ds$

$$\phi = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 I}{2\pi}\int_s^{s+a} \frac{1}{s}\, a\, ds = \frac{\mu_0 I a}{2\pi} \ln s \Big|_s^{s+a}$$

$$= \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{s}\right)$$

b) $$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{\mu_0 I a}{2\pi}\frac{d}{dt}\ln\left(1+\frac{a}{s}\right) = -\frac{\mu_0 I a}{2\pi}\,\frac{-\frac{a}{s^2}}{1+\frac{a}{s}}\,\frac{ds}{dt}$$ $\rightarrow v$

$$= \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

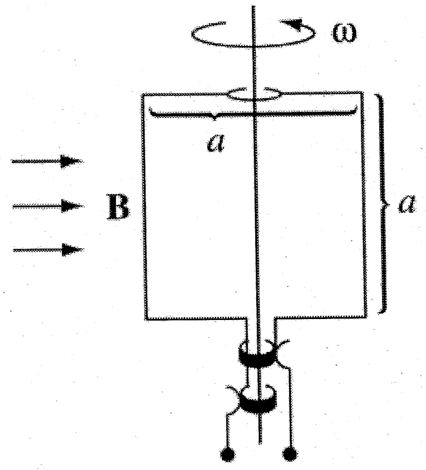
the force on a positive charge in the near by side is to the right $q\,\vec{v}\times\vec{B}$, so driving a current counter clockwise.

c) Now the flux is not changing over the area of the loop (i.e $\phi = BA =$ constant)

$$\Rightarrow \mathcal{E} = -\frac{d\phi}{dt} = 0$$

Problem 7.10 A square loop (side a) is mounted on a vertical shaft and rotated at angular velocity ω (Fig. 7.19). A uniform magnetic field **B** points to the right. Find the $\mathcal{E}(t)$ for this **alternating current** generator.

$$\phi = \vec{B} \cdot \vec{A} = BA \cos\theta = Ba^2 \cos\theta$$



here the angular Position θ

is $\theta = \omega t$

So $\phi = Ba^2 \cos \omega t$

$$\Rightarrow \mathcal{E} = -\frac{d\phi}{dt} = -Ba^2\omega(-\sin\omega t)$$

$$= Ba^2 \omega \sin \omega t$$