Electromagnetic theory (2)

HW # 11 - Solution

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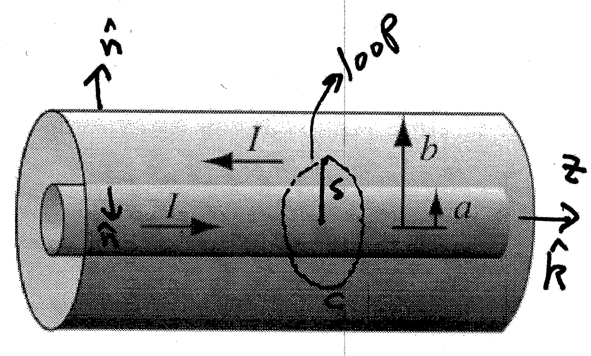
Problem 6.16 : A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. A current **I** flows down the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface (Fig. 6.24). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.



using $\vec{B} = \mu_0(1+\chi_m)\vec{H}$, we need to find $\vec{H}$ first $\Rightarrow \oint_C \vec{H}\cdot d\vec{l} = I^f_{enc} = I$

$\Rightarrow H(2\pi s) = I \Rightarrow \vec{H} = \frac{I}{2\pi s}\hat{\phi}$

$\Rightarrow \vec{B} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi}$

now to check this, let us find $\vec{B}$ in different way;

$\vec{M} = \chi_m \vec{H} = \frac{\chi_m I}{2\pi s}\hat{\phi} \Rightarrow \vec{J}_b = \nabla\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(s M_\phi)\hat{k} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right) = 0$

and $\vec{K}_b = \vec{M}\times\hat{n} = \frac{\chi_m I}{2\pi s}\hat{\phi}\times\hat{n} = \begin{cases} \frac{\chi_m I}{2\pi a}\hat{k}, & \text{at } s=a \text{ where } \hat{n} = -\hat{s} \\ -\frac{\chi_m I}{2\pi b}\hat{k}, & \text{at } s=b, \text{ where } \hat{n} = \hat{s} \end{cases}$

the total enclosed current by the above loop is $I_{enc} = I + \int K_b\, dl = I + K_b l = I + \frac{\chi_m I}{2\pi a}2\pi a$
$= I(1+\chi_m)$

$\Rightarrow \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$B(2\pi s) = \mu_0 I(1+\chi_m) \Rightarrow \vec{B} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi}$ ✓

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let us check B.Cs on $\vec{H}$ and $\vec{B}$ at $s=a$

H: i) $H^\perp_{out} - H^\perp_{in} = -(M^\perp_{out} - M^\perp_{in})$. Here H_{out} is the region $s<a$ and

$M^\perp_{out} = 0$ and $M^\perp_{in} = 0$ as $\vec{M}$ points along the $\hat{\phi}$-direction

$\Rightarrow H^\perp_{out} = H^\perp_{in}$
$0 = 0$ ✓

ii) $H^\parallel_{out} - H^\parallel_{in} = \vec{K}_f\times\hat{n}$

$0 - \frac{I}{2\pi a}\hat{\phi} \overset{?}{=} \frac{I}{2\pi a}\hat{k}\times(-\hat{s})$
$= -\frac{I}{2\pi a}\hat{\phi}$ ✓

$\vec{B}$: at $r=a$ i) $B^\perp_{out} = B^\perp_{in}$
$0 = 0$ ✓

ii) $B^\parallel_{out} - B^\parallel_{in} = \mu_0(\vec{K}_f + \vec{K}_b)\times\hat{n}$

$0 - \mu_0(1+\chi_m)\frac{I}{2\pi a}\hat{\phi} = \mu_0\left(\frac{I}{2\pi a}\hat{k} + \frac{\chi_m I}{2\pi a}\hat{k}\right)\times\hat{n}$

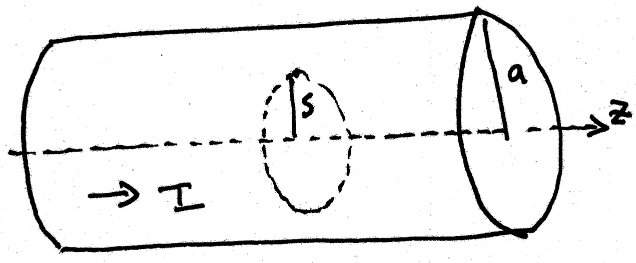
$-\mu_0(1+\chi_m)\frac{I}{2\pi a}\hat{\phi} \overset{?}{=} \mu_0(1+\chi_m)\frac{I}{2\pi a}\hat{k}\times(-\hat{s})$

$= -\mu_0(1+\chi_m)\frac{I}{2\pi a}\hat{\phi}$ ✓

Problem 6.17 A current **I** flows down a long straight wire of radius a. If the wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the net bound current flowing down the wire?

- outside $(s>a)$; $\oint \vec{H}\cdot d\vec{l} = I_{f\,enc}$

$$H(2\pi s) = I \Rightarrow \vec{H} = \frac{I}{2\pi s}\hat{\phi}$$



- inside $I_{enc} = \int J\,da = \frac{I}{\pi a^2}\int da$

$$= \frac{I}{\pi a^2}\pi s^2 = \frac{I s^2}{a^2}$$

so $H(2\pi s) = \frac{I s^2}{a^2} \Rightarrow \vec{H} = \frac{I s}{2\pi a^2}\hat{\phi}$

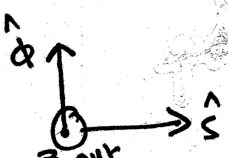
$$\vec{B} = \mu\vec{H} = \begin{cases} \frac{\mu_0(1+\chi_m) I s}{2\pi a^2}\hat{\phi} & ; s<a \\ \frac{\mu_0 I}{2\pi s}\hat{\phi} & ; s>a \end{cases}$$

; here $\chi_m = 0$ outside

now for linear medium $\vec{J}_b = \nabla\times\vec{M} = \nabla\times(\chi_m\vec{H}) = \chi_m\nabla\times\vec{H}$

$= \chi_m\vec{J}_f$, where $J_f = \frac{I}{\pi a^2}$

$= \frac{\chi_m I}{\pi a^2}\hat{k}$



and $\vec{K}_b = \vec{M}\times\hat{n}\Big|_{s=a} = \chi_m\vec{H}\times\hat{n}\Big|_{s=a}$

at $s=a$
$H_{in}(s=a) = H_{out}(s=a)$

$= \frac{\chi_m I}{2\pi a}(\hat{\phi}\times\hat{s}) = \frac{\chi_m I}{2\pi a}(-\hat{k}) = -\frac{\chi_m I}{2\pi a}\hat{k}$

opposite to I

so the total bound current is

$$I_b = \int J_b\,da + \int K_b\,dl$$

$$= \frac{\chi_m I}{\pi a^2}\int da - \frac{\chi_m I}{2\pi a}\int dl$$

$$= \frac{\chi_m I}{\pi a^2}\pi a^2\hat{k} - \frac{\chi_m I}{2\pi a}2\pi a\hat{k} = \chi_m I\hat{k} - \chi_m I\hat{k} = 0$$

as expected. as always the total bound current $= 0$

Problem 6.18: A sphere of linear magnetic material is placed in an otherwise uniform magnetic field B0. Find the new field inside the sphere. [Hint: See Prob. 6.15 or Prob. 4.23.]

There is no free current ($J_f = 0$) $\Rightarrow \nabla \times \vec{H} = 0$

$\Rightarrow \vec{H} = -\nabla \Phi_m \Rightarrow \nabla^2 \Phi_m = 0$, Laplace eq^n

inside the sphere, the solution of Laplace eq^n is given by

$$\Phi_{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta) \quad \text{--- (1)}$$

now the solution outside the sphere must satisfy the B.Cs that for large r

$\vec{B}(r,\theta) \rightarrow \vec{B_0} = B_0 \hat{k}$, $\Rightarrow \vec{B}_{out} = \mu_0 \vec{H}_{out} \Rightarrow \vec{H}_{out} = \dfrac{\vec{B}_{out}}{\mu_0}$

$\Rightarrow \vec{H}_{out} = \dfrac{1}{\mu_0} B_0 \hat{k}$, but $\vec{H}_{out} = -\nabla\phi_{out}(r,\theta)$

$$\Rightarrow \vec{H}_{out} = \frac{1}{\mu_0} B_0 \underset{\downarrow}{\underline{\underline{\hat{k}}}} = -\left[\hat{r}\frac{\partial\phi_{out}}{\partial r} + \frac{1}{r}\frac{\partial\phi_{out}}{\partial\theta}\hat{\theta} + \overset{\text{zero}}{\cancel{\frac{1}{r\sin\theta}\frac{\partial\phi_{out}}{\partial\phi}\hat{\phi}}}\right]$$

$$\Rightarrow \frac{1}{\mu_0} B_0 (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) = -\hat{r}\frac{\partial\phi_{out}}{\partial r} - \frac{1}{r}\frac{\partial\phi_{out}}{\partial\theta}\hat{\theta}$$

equating coeff of unit vectors $\hat{r}$ and $\hat{\theta}$, we get

$\dfrac{\partial\phi_{out}}{\partial r} = -\dfrac{B_0}{\mu_0}\cos\theta$, integrate $\Rightarrow \phi_{out}(r,\theta) = -\dfrac{B_0}{\mu_0} r\cos\theta$

or $\dfrac{1}{r}\dfrac{\partial\phi_{out}}{\partial\theta} = \dfrac{B_0}{\mu_0}\sin\theta \overset{\text{integrate}}{\Longrightarrow} \phi_{out}(r,\theta) = -\dfrac{B_0}{\mu_0} r\cos\theta$ same

so far away from the sphere, the solution is

$$\phi_{out}(r,\theta) = -\frac{B_0}{\mu_0} r\cos\theta$$

so outside the full solution of Laplace eq^n can be written as $\Phi_{out}(r,\theta) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

now let us apply B.Cs

(i) here we can't use $H^{\perp}_{out} - H^{\perp}_{in} = -(M^{\perp}_{out} - M^{\perp}_{in})$ as we have no information about M, so we can alternatively use $B^{\perp}_{out} - B^{\perp}_{in} = 0$, so we get

$$\mu_0 H^{\perp}_{out} - \mu H^{\perp}_{in} = 0 \Rightarrow \mu_0\left(-\frac{\partial \Phi_{out}}{\partial r}\right)\Big|_{r=R} - \mu\left(-\frac{\partial \Phi_{in}}{\partial r}\right)\Big|_{r=R} = 0$$

$$\Rightarrow -\mu_0 \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=R} + \mu \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=R} = 0$$

$$\Rightarrow -\mu_0\left\{-\frac{B_0}{\mu_0}\cos\theta + \sum_l B_l P_l \frac{(l+1)}{r^{l+2}}\right\} + \mu \sum_l A_l l r^{l-1} P_l = 0$$

$$\Rightarrow \sum_l \left[\mu A_l l r^{l-1} - \mu_0 \frac{B_l(l+1)}{r^{l+2}}\right] P_l(\cos\theta) = B_0 \underbrace{\cos\theta}_{P_1}$$

from the structure of the last eq^n, it is obvious that the allowed values of l is one value $(l=1)$, so

$$\left.\begin{aligned} \Phi_{in}(r,\theta) &= A_1 r\cos\theta \\ \Phi_{out}(r,\theta) &= -\frac{B_0}{\mu_0} r\cos\theta + \frac{B_1}{r^2}\cos\theta \end{aligned}\right\} \text{need to find } A_1 \text{ and } B_1$$

now apply again B.Cs when $l=1$

(i) $\phi_{in}(R,\theta) = \phi_{out}(R,\theta) \Rightarrow A_1 R\cos\theta = -\frac{B_0}{\mu_0} R\cos\theta + \frac{B_1}{R^2}\cos\theta$

$$\Rightarrow A_1 R = -\frac{B_0 R}{\mu_0} + \frac{B_1}{R^2} \longrightarrow \boxed{A_1 = -\frac{B_0}{\mu_0} + \frac{B_1}{R^3}} \quad -(1)$$

(ii) $B^{\perp}_{out}\Big|_{r=R} = B^{\perp}_{in}\Big|_{r=R} \Rightarrow \mu_0 \frac{\partial \phi_{out}}{\partial r}\Big|_{r=R} = \mu \frac{\partial \phi_{in}}{\partial r}\Big|_{r=R}$

$$\Rightarrow \mu_0 \left[-\frac{B_0}{\mu_0} \cancel{\cos\theta} - \frac{2B_1}{R^3} \cancel{\cos\theta} \right] = \mu A_1 \cancel{\cos\theta}$$

$$\Rightarrow -B_0 - \frac{2B_1\mu_0}{R^3} = \mu A_1 \Rightarrow \boxed{A_1 = -\frac{B_0}{\mu} - \frac{2B_1\mu_0}{\mu R^3}} \quad -(2)$$

$\Rightarrow$ equate $1 = 2 \Rightarrow -\frac{B_0}{\mu} - \frac{2B_1\mu_0}{\mu R^3} = -\frac{B_0}{\mu_0} + \frac{B_1}{R^3}$

$$\Rightarrow B_0\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) = \frac{B_1}{R^3}\left(1 + \frac{2\mu_0}{\mu}\right) \Rightarrow$$

$$B_0 \frac{\mu - \mu_0}{\mu_0 \cancel{\mu}} = \frac{B_1}{R^3}\left(\frac{\mu + 2\mu_0}{\cancel{\mu}}\right) \Rightarrow B_1 = \frac{B_0 R^3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)$$

$$\Rightarrow A_1 = -\frac{B_0}{\mu_0} + \frac{B_0}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right) = \frac{B_0}{\mu_0}\left[\frac{\mu - \mu_0}{\mu + 2\mu_0} - 1\right]$$

$$= \frac{B_0}{\mu_0}\left[\frac{-3\mu_0}{\mu + 2\mu_0}\right] = \frac{-3B_0}{\mu + 2\mu_0}$$

$$\Rightarrow \phi_{in}(r,\theta) = A_1 r\cos\theta = \frac{-3B_0}{(\mu+2\mu_0)} r\cos\theta \quad , \quad r<R$$

and

$$\phi_{out}(r,\theta) = -\frac{B_0}{\mu_0} r\cos\theta + \frac{B_0}{\mu_0}\frac{R^3}{r^2}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\cos\theta , \quad r>R$$

Now $\vec{H}_{in} = -\nabla\phi_{in} = -\left\{\hat{r}\,\frac{\partial\phi_{in}}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial\phi_{in}}{\partial\theta}\right\}$

$$= -\left[\hat{r}\,A_1\cos\theta - \hat{\theta}\,\frac{1}{\cancel{r}}A_1\cancel{r}\sin\theta\right]$$

$$= -A_1\{\underbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}_{\hat{k}}\} = -A_1\hat{k}$$

$$= \frac{3B_0}{\mu+2\mu_0}\hat{k} \quad , \quad \vec{B}_{in} = \mu\vec{H}_{in} = \frac{3\mu B_0}{\mu+2\mu_0}\hat{k}$$

recall that $\mu = \mu_0(1+\chi_m)$; χ_m : magnetic susceptibility

μ: permeability; μ_0: permeability of free space

$$\Rightarrow \vec{B}_{in} = \frac{3\,\mu/\mu_0\,B_0}{\frac{\mu}{\mu_0}+2}\hat{k} = \frac{3(1+\chi_m)B_0}{1+\chi_m+2}\hat{k}$$

$$= \frac{3(1+\chi_m)B_0\,\hat{k}}{3+\chi_m} = \frac{(1+\chi_m)B_0}{1+\chi_m/3}\hat{k}$$

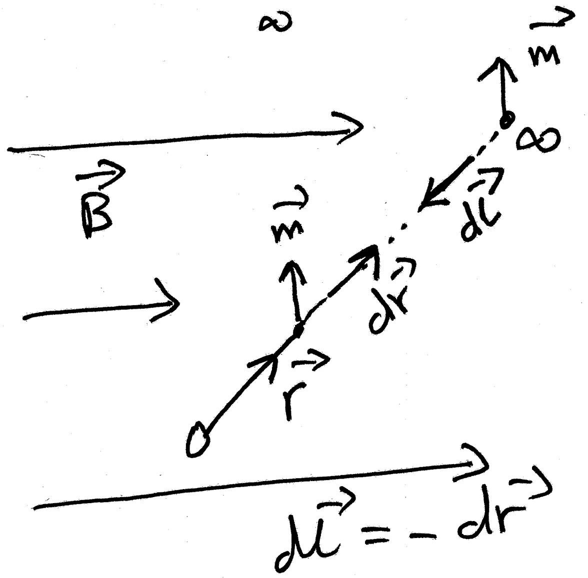
Problem 6.21

(a) Show that the energy of a magnetic dipole in a magnetic field B is U = −m · B.

<u>two methods</u>

(i) The magnetic force exerted by the field $\vec{B}$ on a dipole $\vec{m}$ is $\vec{F} = \nabla(\vec{m}\cdot\vec{B})$. Now to move a dipole from ∞ to a position $\vec{r}$ inside the field (with a constant speed), one must exert an opposite force of $-\nabla(\vec{m}\cdot\vec{B})$, so the change in potential energy is

$$\Delta U = -\int_{\infty}^{\vec{r}} \vec{F}\cdot d\vec{l} = -\int_{\infty}^{\vec{r}} -\nabla(\vec{m}\cdot\vec{B})\cdot d\vec{l} = \int_{\infty}^{\vec{r}} \nabla(\vec{m}\cdot\vec{B})\cdot d\vec{l}$$



$$= -\int_{\infty}^{\vec{r}} \nabla(\vec{m}\cdot\vec{B})\cdot d\vec{r}$$

Now using the gradient theorem eq^n 1.55

$$U(\vec{r}) - \underbrace{U(\infty)}_{0} = -[\vec{m}\cdot\vec{B}(\vec{r}) - \vec{m}\cdot\underbrace{\vec{B}(\infty)}_{\text{zero}}]$$

so as long as $\vec{B} = 0$ at ∞, then

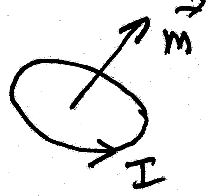
$$U = -\vec{m}\cdot\vec{B}$$

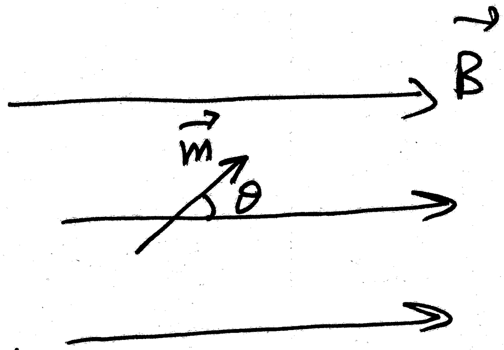
Now let us derive the same result using the traditional method given in the next page

(ii) method 2

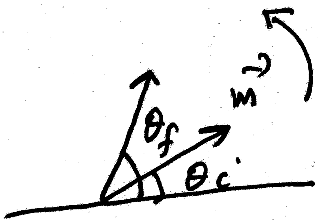
show that the energy of a magnetic dipole in a magnetic field $\vec{B}$ is given by $U = -\vec{m}\cdot\vec{B}$

- consider a dipole $\vec{m}$ placed in a uniform magnetic field as shown in figure. the dipole could be a current loop for example





the dipole prefers to be oriented with the direction of the field. consider an external agent that rotates the dipole from θ_i to θ_f then the work done by this agent to rotate the dipole by an infinitismal angle $d\theta$ is



$$dw_{ext} = \vec{N}\, d\theta = \vec{m}\times\vec{B}\, d\theta = mB\sin\theta\, d\theta$$

$$W_{ext} = \int_{\theta_i}^{\theta_f} mB\sin\theta\, d\theta = mB(\cos\theta_i - \cos\theta_f)$$

this work is stored into the system as a magnetic potential energy, so $W_{ext} = \Delta U$

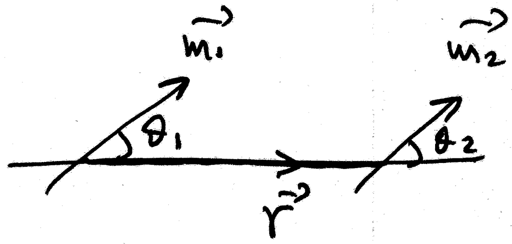
$$\Rightarrow \Delta U = mB(\cos\theta_i - \cos\theta_f)$$
$$= mB\cos\theta_i - mB\cos\theta_f$$
$$= -mB\cos\theta_f$$

, choosing $U_i = 0$ at $\theta = \pi/2$ as a reference

$\Rightarrow$ so $\quad W_f = W = -mB\cos\theta$

$$= -\vec{m}\cdot\vec{B} \quad ✓$$

Now what is the interaction energy between two dipoles separated by a distance r as shown in figure



$$\vec{B}_{1\,dip} = \frac{\mu_0}{4\pi r^3}\left(3(\vec{m_1}\cdot\hat{r})\hat{r} - \vec{m_1}\right)$$

so $W = -\vec{m_2}\cdot\vec{B_1}$

$$= -\frac{\mu_0}{4\pi r^3}\left(3(\vec{m_1}\cdot\hat{r})(\vec{m_2}\cdot\hat{r}) - \vec{m_1}\cdot\vec{m_2}\right)$$

$$= \frac{\mu_0}{4\pi r^3}\left(\vec{m_1}\cdot\vec{m_2} - 3(\vec{m_1}\cdot\hat{r})(\vec{m_2}\cdot\hat{r})\right)$$

this can be written in terms of θ_1 and θ_2 as

$$W = \frac{\mu_0}{4\pi r^3}\left[m_1 m_2\cos(\theta_2-\theta_1) - 3m_1\cos\theta_1\, m_2\cos\theta_2\right]$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 - 3\cos\theta_1\cos\theta_2\right]$$

$$= \frac{\mu_0 m_1 m_2}{4\pi r^3}\left[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right]$$

the stable angular positions of $\vec{m_1}$ and $\vec{m_2}$ occur when $\frac{\partial W}{\partial\theta_1} = \frac{\partial W}{\partial\theta_2} = 0$, so

$$\frac{\partial W}{\partial\theta_1} = \frac{\mu_0 m_1 m_2}{4\pi r^3}\left(\cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2\right) = 0 \quad — (1)$$

and

$$\frac{\partial W}{\partial\theta_2} = \frac{\mu_0 m_1 m_2}{4\pi r^3}\left(\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2\right) = 0 \quad — (2)$$

so $$2\sin\theta_1\cos\theta_2 = -\cos\theta_1\sin\theta_2 \quad \text{---} (3)$$

and $$\sin\theta_1\cos\theta_2 = -2\cos\theta_1\sin\theta_2 \quad \text{---} (4)$$

substitute (4) into (3), we get

$$-4\cos\theta_1\sin\theta_2 = -\cos\theta_1\sin\theta_2 \Rightarrow -3\cos\theta_1\sin\theta_2 = 0$$

$$\Rightarrow \boxed{\cos\theta_1\sin\theta_2 = 0}$$

Now substitute (3) into (4)

we get

$$\sin\theta_1\cos\theta_2 = 4\sin\theta_1\cos\theta_2$$

$$\Rightarrow 3\sin\theta_1\cos\theta_2 = 0 \Rightarrow \boxed{\sin\theta_1\cos\theta_2 = 0}$$

Thus $$\boxed{\sin\theta_1\cos\theta_2 = \sin\theta_2\cos\theta_1 = 0}$$

two possibilities

either $\sin\theta_1 = \sin\theta_2 = 0$ $\qquad \overset{\vec{m_1}}{\longrightarrow} \; \overset{\vec{m_2}}{\longrightarrow}$ ($\theta_1 = 0$, $\theta_2 = 0$) or $\overset{\vec{m_1}}{\longrightarrow} \; \overset{\vec{m_2}}{\longleftarrow}$ ($\theta_1 = 0$, $\theta_2 = \pi$)

or $\cos\theta_1 = \cos\theta_2 = 0$ $\qquad \vec{m_1}\uparrow \; \uparrow\vec{m_2}$ ($\theta_1 = \pi/2$, $\theta_2 = \pi/2$) or $\overset{\vec{m_1}}{\uparrow} \; \overset{\vec{m_2}}{\downarrow}$ ($\theta_1 = \pi/2$, $\theta_2 = -\pi/2$)

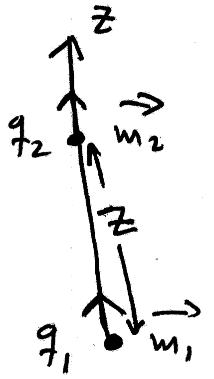
Now which one is the most stable configuration. This can be determined by finding W for each configuration, and hence the lowest energy configuration will be the most stable one

θ_1	θ_2	$W\left(\frac{\mu_0 m_1 m_2}{4\pi r^3}\right)$
0	0	-2 ✓
0	π	$+2$
$\pi/2$	$\pi/2$	1
$\pi/2$	$-\pi/2$	-1

$\Rightarrow$ this is the most stable configuration, where they both line up parallel along the line joining them

$$\overset{}{\underset{\vec{m_1}}{\longrightarrow}} \qquad \underset{\vec{m_2}}{\longrightarrow}$$

Problem 6.24: Imagine two charged magnetic dipoles (charge q, dipole moment m), constrained to move on the z axis (same as Problem 6.23(a), but without gravity). Electrically they repel, but magnetically (if both m's point in the z direction) they attract. (a) Find the equilibrium separation distance. (b) What is the equilibrium separation for two electrons in this orientation. [Answer: 4.72 × 10−13 m.] (c) Does there exist, then, a stable bound state of two electrons?



a) $q_1 = q_2 = q$ and $m_1 = m_2 = m \Rightarrow \vec{m} = m\hat{k}$

the electric force on the upper charge is

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{z^2}\hat{k} \quad \text{---(1) upward}$$

Now need to find the magnetic force exerted by $\vec{m}_1$ on $\vec{m}_2$. first find $\vec{B}_1$ at the location of $\vec{m}_2$.

$\vec{B}_1 = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$, but $r = z$, $\hat{r} = \hat{k}$, $\theta = 0$, so

$= \frac{\mu_0 m}{2\pi z^3}\hat{k}$, Now the magnetic force on the upper dipole is $\vec{F}_m = \nabla(\vec{m}_2 \cdot \vec{B}_1) = \nabla\left(m\hat{k}\cdot\frac{\mu_0 m}{2\pi z^3}\hat{k}\right) = \frac{\mu_0 m^2}{2\pi}\frac{d}{dz}\left(\frac{1}{z^3}\right)\hat{k}$

$$= -\frac{3\mu_0 m^2}{2\pi z^4}\hat{k} \quad \text{downward ---(2)}$$

at equilibrium $|\vec{F}_e| = |\vec{F}_m| \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{q^2}{z^2} = \frac{3\mu_0 m^2}{2\pi z^4}$

$$\Rightarrow z = \sqrt{6}\sqrt{\mu_0\varepsilon_0}\,\frac{m}{q} = \sqrt{6}\,\frac{m}{qc}$$

b) for electron $q = 1.6\times10^{-19}\,C$, $m = \mu_B = 9.22\times10^{-24}\,Am^2$, $c = 3\times10^8\,m/s$

$$\Rightarrow z = 4.7\times10^{-13}\,m = 470\,pm$$

c) No. This is an unstable equilibrium.

Problem 6.26 Compare Eqs. 2.15, 4.9, and 6.11. Notice that if ρ, $\mathbf{P}$, and $\mathbf{M}$ are *uniform*, the *same integral* is involved in all three:

$$\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'.$$

Therefore, if you happen to know the electric field of a uniformly *charged* object, you can immediately write down the scalar potential of a uniformly *polarized* object, and the vector potential of a uniformly *magnetized* object, of the same shape. Use this observation to obtain V inside and outside a uniformly polarized sphere (Ex. 4.2), and $\mathbf{A}$ inside and outside a uniformly magnetized sphere (Ex. 6.1).

equation 2.15 ; $\vec{E} = \rho \left\{ \frac{1}{4\pi\varepsilon_0} \int_V \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \right\}$ for uniform charge density

equation 4.9 ; $V = \vec{P} \cdot \left\{ \frac{1}{4\pi\varepsilon_0} \int_V \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \right\}$ for uniform polarization

equation 6.11 ; $\vec{A} = \mu_0 \varepsilon_0 \vec{M} \times \left\{ \frac{1}{4\pi\varepsilon_0} \int_V \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \right\}$ for uniform magnetization

so from 2.15 the integral

$\frac{1}{4\pi\varepsilon_0} \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' = \frac{\vec{E}}{\rho}$; for uniformally charged sphere, we have

$$\begin{cases} \vec{E}_{in} = \rho\left(\frac{1}{3\varepsilon_0}\vec{r}\right) & \text{; (problem 2.12)} \\ \vec{E}_{out} = \rho\left(\frac{1}{3\varepsilon_0}\frac{R^3}{r^2}\hat{r}\right) & \text{; (example 2.3)} \end{cases}$$

So once we know $\vec{E}$, we know both V and $\vec{A}$

=> So we can then obtain

$$\begin{cases} V_{in} = \vec{P} \cdot \frac{\vec{E}_{in}}{\rho} = \frac{1}{3\varepsilon_0}(\vec{P}\cdot\vec{r}) \\ V_{out} = \vec{P} \cdot \frac{\vec{E}_{out}}{\rho} = \frac{1}{3\varepsilon_0}\frac{R^3}{r^2}(\vec{P}\cdot\hat{r}) \end{cases}$$

and finally the vector potential A

$$\begin{cases} \vec{A}_{in} = \mu_0\varepsilon_0 \vec{M} \times \frac{\vec{E}_{in}}{\rho} = \frac{\mu_0}{3}(\vec{M}\times\vec{r}) \\ \vec{A}_{out} = \mu_0\varepsilon_0 \vec{M} \times \frac{\vec{E}_{out}}{\rho} = \frac{\mu_0}{3}\frac{R^3}{r^2}(\vec{M}\times\hat{r}) \end{cases}$$