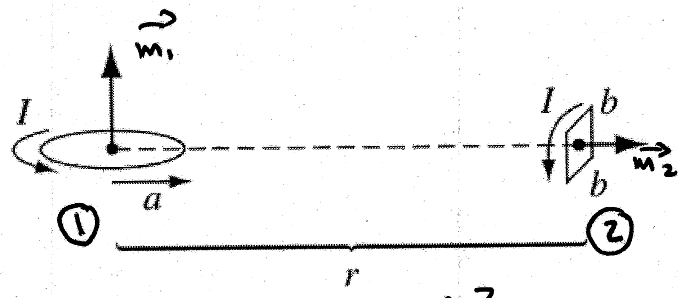
Electromagnetic theory (2)

HW # 10 - solution

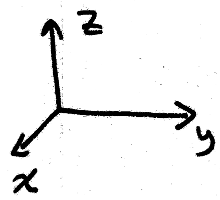
Dr. Gassem Alzoubi

Problem 6.1 Calculate the torque exerted on the square loop shown in figure, due to the circular loop (assume r is much larger than *a* or *b*). If the square loop is free to rotate, what will its equilibrium orientation be?



$\vec{m}_1 = m_1\hat{k}$; $\vec{m}_2 = m_2\hat{j}$; $\vec{r} = r\hat{j}$

$\vec{m}_1 = \pi a^2 I\hat{k}$; $\vec{m}_2 = Ib^2\hat{j}$

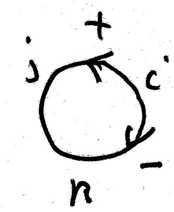


recall that the magnetic field created by a dipole at $\vec{r}$ is given by

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi r^3}\left(3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right)$$

so the field created by the circular loop at the position of the square loop is

$$\vec{B}_1 = \frac{\mu_0}{4\pi r^3}\left(3(m_1\underbrace{\hat{k}\cdot\hat{r}}_{\substack{\text{zero}\\ \theta=90}}) - m_1\hat{k}\right) = \frac{-\mu_0 m_1}{4\pi r^3}\hat{k}$$



so $\vec{N}_2 = \vec{m}_2 \times \vec{B}_1 = m_2\hat{j} \times \left(-\frac{\mu_0 m_1}{4\pi r^3}\hat{k}\right) = -\frac{\mu_0 m_1 m_2}{4\pi r^3}\hat{i}$

$$= \frac{-\mu_0}{4\pi r^3}(\pi a^2 I)(Ib^2)\hat{i} = -\frac{\mu_0 (Iab)^2}{4r^3}\hat{i}$$

This torque will rotate $\vec{m}_2$ to make it alligned with the direction of $\vec{B}_1$, so the final equilibrium position of $\vec{m}_2$ is when it points in the negative z-direction. once reachs there, the torque on $\vec{m}_2$ will become zero as ($\vec{N}_2 = \vec{m}_2 \times \vec{B}_1 = 0$); since $\vec{m}_2$ and $\vec{B}_1$ are now parallel, so their cross product is zero.

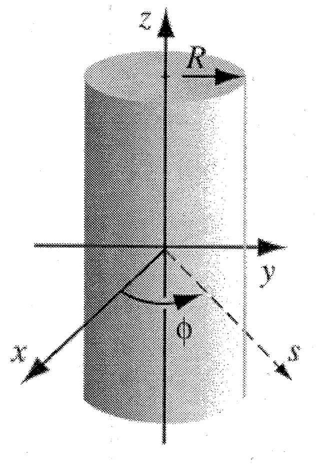
this is consistant with the fact that the energy of the dipole will be minimized when both $\vec{m}_2$ and $\vec{B}_1$ are parallel ; $U_{mag} = -\vec{m}_2 \cdot \vec{B}_1 = -m_2 B_1 \cos\theta$

$$= \begin{cases} -m_2 B_1, & \theta = 0, \text{ parallel} \\ m_2 B_1, & \theta = 180, \text{ antiparallel} \end{cases}$$

Problem 6.8: A long circular cylinder of radius R carries a magnetization $\vec{M} = ks^2\,\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector (Fig. 6.13). Find the magnetic field due to $\vec{M}$, for points inside and outside the cylinder.

in cylindrical coordinates, where $h_1 = 1,\ h_2 = s,\ h_3 = 1$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{s} & h_2\hat{\phi} & h_3\hat{k} \\ \partial/\partial s & \partial/\partial\phi & \partial/\partial z \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \quad ; \quad \begin{aligned} A_1 &= A_s \\ A_2 &= A_\phi \\ A_3 &= A_z \end{aligned}$$



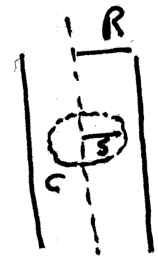
$$\text{so } \vec{J}_b = \nabla\times\vec{M} = \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{k} \\ \partial/\partial s & \partial/\partial\phi & \partial/\partial z \\ 0 & s(ks^2) & 0 \end{vmatrix} = \frac{1}{s}\frac{\partial}{\partial s}(ks^3)\,\hat{k} = 3ks\,\hat{k}$$

$$\text{and } \vec{K}_b = \vec{M}\times\hat{n}\Big|_{s=R} = kR^2(\hat{\phi}\times\hat{s}) = kR^2(-\hat{k}) = -kR^2\,\hat{k}$$

so the J_b flows up and then returns as a surface current (down), so the net bound currents is zero. let us check that;

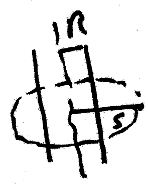
$$I = \int J_b\,da + \int K_b\,dl = \int_0^R 3ks\,(2\pi s\,ds) + \int(-kR^2)\,dl = 2\pi kR^3 - kR^2(2\pi R) = \text{zero}$$

now using Ampere's law, we get the field inside,



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B(2\pi s) = \mu_0\int_0^s J_b\,da = 2\pi k\mu_0 s^3$$

$$\Rightarrow \vec{B}_{in} = \mu_0 k s^2\,\hat{\phi}\ ;$$



$$\text{outside } \oint \vec{B}_{out}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\text{but } I_{enc} = \text{zero} \Rightarrow \vec{B}_{out} = 0$$

Note: B.Cs of $\vec{B}$ are satisfied on the surface

i) $B^{\perp}_{in} = B^{\perp}_{out}$

$0 = 0$ ✓

; ii) $B^{\parallel}_{out} - B^{\parallel}_{in} = \mu_0\,\vec{K}_b\times\hat{n}$

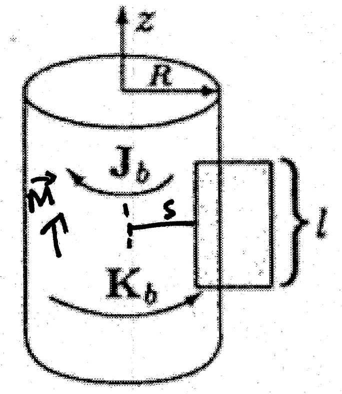
$$0 - \mu_0 kR^2\,\hat{\phi} \stackrel{?}{=} \mu_0(-kR^2\,\hat{k}\times\hat{s})$$

$$= -\mu_0 kR^2\,\hat{\phi}\ ✓$$

Problem 6.12 An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis, $M = ks\,\hat{k}$, where k is a constant and s is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

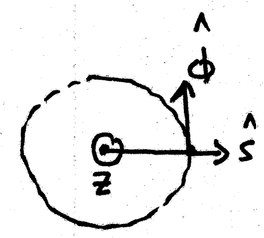
(a) As in Sect. 6.2, locate all the bound currents, and calculate the field they produce. **(b)** Use Ampère's law (in the form of Eq. 6.20) to find H, and then get B from Eq. 6.18. (Notice that the second method is much faster, and avoids any explicit reference to the bound currents.)

a) $\vec{J}_b = \nabla \times \vec{M} = \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{k} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & ks \end{vmatrix} = -k\hat{\phi}$

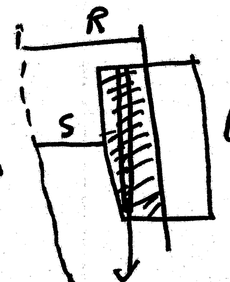


$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{s}\Big|_{s=R} = ks\,(\hat{k}\times\hat{s})\Big|_{s=R} = kR\hat{\phi}$

it can be considered as a superposition of two solenoids, so the field outside is zero, and inside will be in the z-direction (same direction of $\vec{M}$).



Now consider the Amperian loop shown in figure



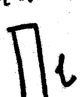
$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$, where $I_{enc} = \int J_b\,da + K_b l$

here we need to consider the currents passing through the shaded area, so

$I_{enc} = \int_s^R (-k)\,l\,ds + K_b l = -kl(R-s) + kRl = kls$

ds, so $da = l\,ds$

(B.Cs): i) $B_{in}^{\perp} = B_{out}^{\perp}$
$0 = 0$
ii) $B_{out}^{\parallel} - B_{in}^{\parallel} = \mu_0 \vec{K}_b \times \hat{n} = \mu_0 kR\,\hat{\phi}\times\hat{s} = -\mu_0 kR\hat{k}$
$0 - \mu_0 kR\hat{k} \overset{?}{=} \mu_0 kR$



$\Rightarrow Bl = \mu_0 kls \Rightarrow \vec{B} = \mu_0 ks\,\hat{k}$

b) Now $\vec{B}$ and $\vec{H}$ should point either parallel or anti parallel to $\vec{M}$. we found in (a) that $\vec{B}$ points along z direction, so does $\vec{H}$. Now applying Ampers' law on $\vec{H}$ using the same loop shown above we get

$\oint \vec{H}\cdot d\vec{l} = I_f = 0 \Rightarrow Hl = 0 \Rightarrow H = 0 \Rightarrow \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = 0$

$\Rightarrow \vec{B}_{in} = \mu_0\vec{M} = \mu_0 ks\,\hat{k}$. outside $M = 0$, so $\vec{B}_{out} = 0$.

we can find H_{in} by taking a circular loop as shown $\Rightarrow \oint_c \vec{H}\cdot d\vec{l} = I_f = 0 \Rightarrow H_{in}(2\pi s) = 0 \Rightarrow H_{in} = 0$, similarly for

$H_{out} = 0$ i.e $H_{in} = H_{out} = 0$

Problem 6.15 :

we found that $\nabla \times \vec{H} = \vec{J}_f$; if $\vec{J}_f = 0$, then $\nabla \times \vec{H} = 0$

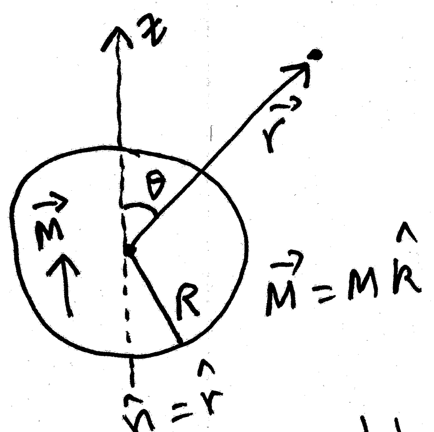
then $\vec{H} = -\nabla \Phi_m$; where Φ_m : scalar magnetic Potential

so $\nabla \times \vec{H} = \nabla \times (-\nabla \Phi_m) = -\nabla^2 \Phi_m = 0 \Rightarrow \boxed{\nabla^2 \Phi_m = 0}$

this is Laplace eq^n and in spherical coordinates the solution takes the form $\Phi_m(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$;

with Azimuthal symmetry. The coefficients A_l, B_l are found using the B.Cs of the problem.

As an example, consider a uniformally magnetized sphere as shown in figure



$\Phi_m^{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$, $r < R$ --- (1)

$\Phi_m^{out}(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$; $r > R$ --- (2)

$\vec{M} = M\hat{k}$

B.Cs:

① $\Phi_{in}(R,\theta) = \Phi_{out}(R,\theta)$ ② $H_{out}^{\perp} - H_{in}^{\perp} = -(\underbrace{M_{out}^{\perp}}_{\text{zero outside}} - M_{in}^{\perp})$

$\Rightarrow -\left.\frac{\partial \Phi_{out}}{\partial r}\right|_{r=R} + \left.\frac{\partial \Phi_{in}}{\partial r}\right|_{r=R} = M_{in}^{\perp}$; but

$\vec{M} = M\hat{k} = M(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$; $M_{in}^{\perp}$ is the normal component i.e $M_{in}^{\perp} = M\cos\theta$

$\Rightarrow \left.\frac{\partial \Phi_{in}}{\partial r}\right|_{r=R} - \left.\frac{\partial \Phi_{out}}{\partial r}\right|_{r=R} = M \underbrace{\cos\theta}_{P_1(\cos\theta)}$ --- (3)

from the R.H.S of eq^n (3), it is clear that $l = 1$ only, so

$\Rightarrow$ so $\Phi_{in} = A_1 r \cos\theta$; $\Phi_{out} = \frac{B_1}{r^2}\cos\theta$

from B.Cs (2), we get $A_1 \cos\theta - B_1 \cos\theta\left(-\frac{2}{R^3}\right) = M \cos\theta$

$\Rightarrow \boxed{A_1 + \frac{2B_1}{R^3} = M}$ --- (4) and from B.Cs (1), we get

$A_1 R \cancel{\cos\theta} = \frac{B_1}{R^2}\cancel{\cos\theta} \Rightarrow \boxed{A_1 = \frac{B_1}{R^3}}$ --- (5)

solving (4) and (5), we get $A_1 = \frac{M}{3}$ and $B_1 = \frac{MR^3}{3}$

$\Rightarrow \Phi_{in}(r,\theta) = \frac{1}{3} M r\cos\theta$; $\Phi_{out}(r,\theta) = \frac{MR^3}{3r^2}\cos\theta$

now $H_{in} = -\nabla\Phi_{in} = -\left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right\}\Phi_{in}$

$$= -\frac{1}{3}M\cos\theta\,\hat{r} - \frac{1}{r}\frac{1}{3}Mr(-\sin\theta)\,\hat{\theta} = -\frac{1}{3}M\underbrace{\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]}_{\hat{k}}$$

$$= -\frac{1}{3}M\hat{k} = -\frac{1}{3}\vec{M}$$

$\Rightarrow \vec{B}_{in} = \mu_0(\vec{H}_{in} + \vec{M}_{in}) = \mu_0\left(-\frac{\vec{M}}{3} + \vec{M}\right) = \frac{2}{3}\mu_0\vec{M}$

and $\vec{H}_{out} = -\nabla\Phi_{out} = -\left[-\frac{2MR^3}{3r^3}\cos\theta\,\hat{r} - \frac{1}{r}\frac{MR^3}{3r^2}(-\sin\theta)\,\hat{\theta}\right]$

$$= \frac{MR^3}{3r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$\Rightarrow \vec{B}_{out} = \mu_0(\vec{H}_{out} + \overset{\text{zero}}{\cancel{\vec{M}_{out}}}) = \frac{\mu_0 MR^3}{3r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$

using $m = \frac{4}{3}\pi R^3 M$ $\quad = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$

equivalent to a single dipole $m = 4/3\,\pi R^3 M$ located at the origin