Electromagnetic theory (1)

HW # 9 - Solution

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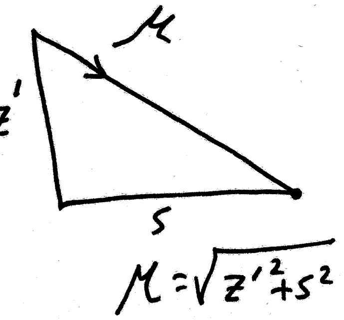
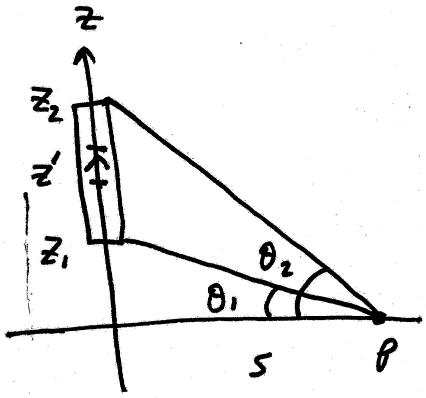
Problem 5.23 Find the magnetic vector potential of a finite segment of straight wire carrying a current **I**. [Put the wire on the z axis, from z_1 to z_2, and use Eq. 5.66.] Check that your answer is consistent with Eq. 5.37.

$d\ell' = dz'$ and $\vec{I} = I\,\hat{k}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,dz'}{\mu}\,\hat{k}$$





$$= \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz'}{\sqrt{z'^2+s^2}}\,\hat{k}$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[z' + \sqrt{z'^2+s^2}\right]_{z_1}^{z_2} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\right]\hat{k}$$

notice that $\vec{A}$ depends on s only; $\vec{A} = \cancelto{0}{A_s}\hat{s} + \cancelto{0}{A_\phi}\hat{\phi} + A_z\hat{k}$
$= A_z\hat{k}$

let us check this result by finding $\vec{B} = \nabla\times\vec{A}$ in cylindrical

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{s} & h_2\hat{\phi} & h_3\hat{k} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial\phi} & \partial/\partial z \\ h_1A_s & h_2A_\phi & h_3A_z \end{vmatrix};\quad \begin{aligned} h_1 &= 1 \\ h_2 &= s \\ h_3 &= 1 \end{aligned}$$

$$= \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{k} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & A \end{vmatrix}$$

$$= \frac{1}{s}\left[\hat{s}\left(\cancelto{0}{\frac{\partial A}{\partial\phi}}\right) - s\hat{\phi}\left(\frac{\partial A}{\partial s}\right) + \hat{k}(0)\right] = -\frac{\partial A}{\partial s}\hat{\phi}$$

$$\Rightarrow \vec{B} = -\frac{\partial A}{\partial s}\hat{\phi}$$

$$= -\frac{\mu_0 I}{4\pi}\frac{\partial}{\partial s}\left[\frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\right]\hat{\phi}$$; derive and simplify, we get

$$= \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - \frac{z_1}{\sqrt{z_1^2+s^2}}\right]\hat{\phi} = \frac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]\hat{\phi}$$

as obtained before using the Biot-Savart law

Problem 5.25 If B is uniform, show that **A**(**r**) = − 1/2(**r** × **B**) works. That is, check that ∇ · **A** = 0 and ∇ × **A** = **B**. Is this result unique, or are there other functions with the same divergence and curl?

first $\nabla \cdot \vec{A} = -\frac{1}{2} \nabla \cdot (\vec{r} \times \vec{B}) = -\frac{1}{2}\left[\vec{B} \cdot (\nabla \times \vec{r}) - \vec{r} \cdot (\nabla \times \vec{B})\right]$

but $\nabla \times \vec{B} = 0$ as B is uniform = constant

and $\nabla \times \vec{r} = 0$ see problem 1.63 in chapter 1

$\Rightarrow \nabla \cdot \vec{A} = 0$

Second

$\nabla \times \vec{A} = -\frac{1}{2} \nabla \times (\vec{r} \times \vec{B})$

$= -\frac{1}{2}\left[(\vec{B} \cdot \nabla)\vec{r} - (\vec{r} \cdot \nabla)\vec{B} + \vec{r}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{r})\right]$

but $(\vec{r} \cdot \nabla)\vec{B}$ and $\nabla \cdot \vec{B}$ are zeros as B is uniform

and $\nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$, and finally

$(\vec{B} \cdot \nabla)\vec{r} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$

$= B_x \hat{i} + B_y \hat{j} + B_z \hat{k} = \vec{B}$

So

$\nabla \times \vec{A} = -\frac{1}{2}\left[(\vec{B} \cdot \nabla)\vec{r} - \vec{B}(\nabla \cdot \vec{r})\right]$

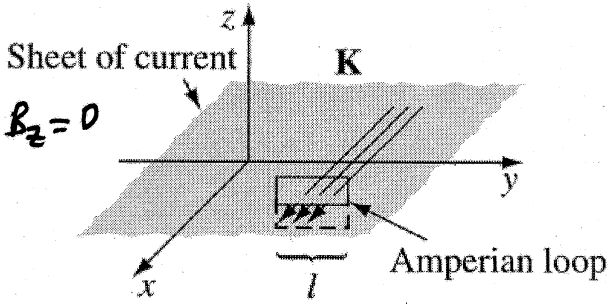
$= -\frac{1}{2}\left[\vec{B} - 3\vec{B}\right] = -\frac{1}{2}(-2\vec{B}) = \vec{B}$ ✓

as expected

Problem 5.27 Find the vector potential above and below the plane surface current in Ex. 5.8. $\vec{K} = K\hat{i}$

from Example 5.8, we found that

$$\vec{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{j} & ; \text{ for } z>0 \\ \frac{\mu_0 K}{2}\hat{j} & ; \text{ for } z<0 \end{cases} \Rightarrow B_x = B_z = 0$$



now from the definition of $\vec{A}$

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mu}\,da'$, we see that $\vec{A}$ is parallel to $\vec{K}$

so using $\vec{B} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A & 0 & 0 \end{vmatrix}$

$$= \frac{\partial A}{\partial z}\hat{j} - \frac{\partial A}{\partial y}\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
$$= 0\,\hat{i} + B_y\hat{j} + 0\,\hat{k}$$

but $\vec{B}$ does not have a component along z direction

$\Rightarrow \frac{\partial A}{\partial y} = 0$, so $B = \frac{\partial A}{\partial z}$

- for $z>0$, we have $-\frac{\mu_0 K}{2} = \frac{\partial A}{\partial z} \Rightarrow A_{above} = -\frac{\mu_0 K}{2}z + C_1$

- for $z<0$, we have $\frac{\mu_0 K}{2} = \frac{\partial A}{\partial z} \Rightarrow A_{below} = \frac{\mu_0 K}{2}z + C_2$

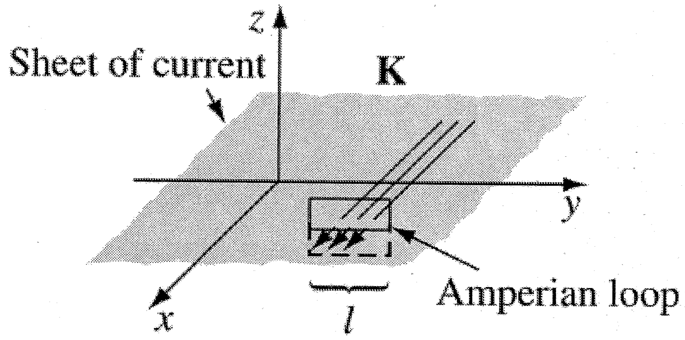
recall that $\vec{A}$ is not uniquely defined, so we are at libre[r]ty to take or choose C_1 and C_2 whatever we want the simplest choice is to take $C_1 = C_2 = 0$, so

$$\vec{A} = \begin{cases} -\frac{\mu_0 K}{2}z\,\hat{i}; & \text{for } z>0 \\ \frac{\mu_0 K}{2}z\,\hat{i}; & \text{for } z<0 \end{cases}$$

another way of solving problem 5.27

Problem 5.27 Find the vector potential above and below the plane surface current in Ex. 5.8.

Ex 5.8 showed that the field is uniform above and Below the sheet and is given by



$$\vec{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{j} & ; \ z>0 \\ \frac{\mu_0 K}{2}\hat{j} & ; \ z<0 \end{cases}$$

since $\vec{B}$ is uniform, we can use the result of P 5.25 to find $\vec{A}$

so for $z>0$, $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B}) = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & -\frac{\mu_0 K}{2} & 0 \end{vmatrix}$

$$= -\frac{\mu_0 K}{4} z\,\hat{i} + \frac{\mu_0 K}{4} x\,\hat{k}$$

we can verify that $\vec{A}$ is correct by finding $\nabla\times\vec{A}$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\mu_0 K}{4}z & 0 & \frac{\mu_0 K}{4}x \end{vmatrix} = -\frac{\mu_0 K}{2}\hat{j} = \vec{B} \text{ for } z>0$$

similar calculations can be done for the case $z<0$; when we get for $\vec{A}$

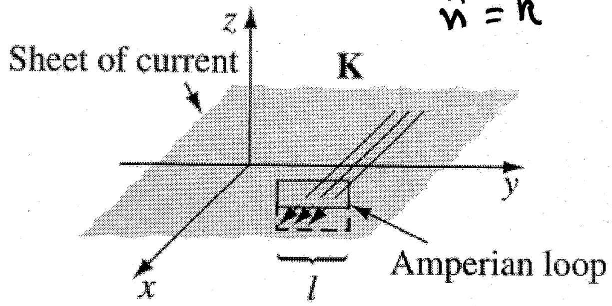
$\vec{A} = \frac{\mu_0 K z}{4}\hat{i} - \frac{\mu_0 K x}{4}\hat{k}$; recall that $\vec{A}$ is Not uniquely defined

Problem verify that the boundary conditions of **A** and **B** are satisfied for problem 5.27

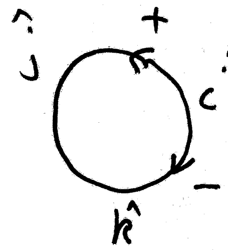
$\vec{K} = K\hat{i}$

$\hat{n} = \hat{k}$

from problem 5.27, we found



$$\vec{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{j} \ ; & z>0 \\ \frac{\mu_0 K}{2}\hat{j} \ ; & z<0 \end{cases}$$



so $\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$

$$-\frac{\mu_0 K}{2}\hat{j} - \frac{\mu_0 K}{2}\hat{j} \stackrel{?}{=} \mu_0 K (\hat{i} \times \hat{k})$$

$$-\mu_0 K \hat{j} \stackrel{?}{=} \mu_0 K(-\hat{j})$$

$$-\mu_0 K \hat{j} = -\mu_0 K \hat{j} \quad \checkmark \Rightarrow \begin{cases} B^{\perp}_{above} = B^{\perp}_{below} = 0 \\ |B^{\parallel}_{above}| = |B^{\parallel}_{below}| \end{cases}$$

and for $\vec{A}$, we found that

$$\vec{A} = \begin{cases} -\frac{\mu_0 K}{2} z\hat{i} \ ; & z>0 \\ \frac{\mu_0 K}{2} z\hat{i} \ ; & z<0 \end{cases} ;$$

notice that $A^{\perp}_{above} = A^{\perp}_{below} = 0$

and $|A^{\parallel}_{above}| = |A^{\parallel}_{below}|$

and

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K} \ ; \quad n = z$$

check

$$\frac{\partial \vec{A}_{above}}{\partial z} - \frac{\partial \vec{A}_{below}}{\partial z} \stackrel{?}{=} -\mu_0 \vec{K}$$

$$-\frac{\mu_0 K}{2}\hat{i} - \frac{\mu_0 K}{2}\hat{i} \stackrel{?}{=} -\mu_0 K \hat{i}$$

$$-\mu_0 K \hat{i} = -\mu_0 K \hat{i} \quad \checkmark$$

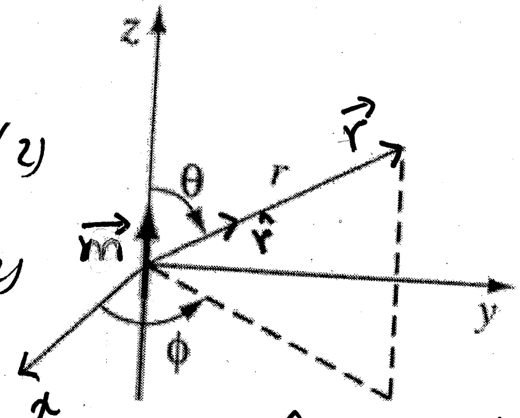
Problem 5.34: Show that the magnetic field of a dipole can be written in coordinate-free form:

$$\boxed{\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}].} \quad \text{---(1)}$$

, let us place $\vec{m}$ at the origin and point along $+z$ direction.

we already found that $B_{dip}(\vec{r})$ is

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] \text{---(2)}$$



we need to show that (1) and (2) are equivalent. Now

$\vec{m} = m\hat{k}$; but in spherical coordinates $\hat{k}$ is given by $\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$\Rightarrow \vec{m} = m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta}$, so starting from (1)

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[3m\cos\theta\,\hat{r} - (m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta})\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[3m\cos\theta\,\hat{r} - m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left\{3m\cos\theta\,\hat{r} \ldots\right.$$

$$= \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$ which is eq^n (2)

so the two forms are equivalent.

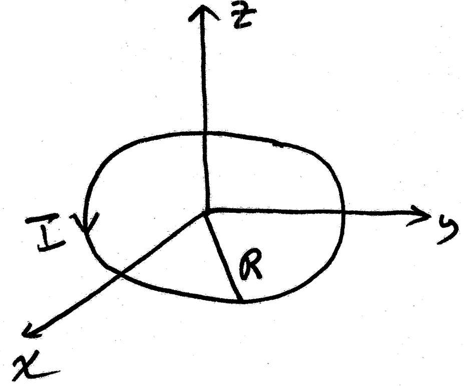
Problem 5.35 A circular loop of wire, with radius R, lies in the xy plane (centered at the origin) and carries a current I running counterclockwise as viewed from the positive z axis.

(a) What is its magnetic dipole moment?

(b) What is the (approximate) magnetic field at points far from the origin?

(c) Show that, for points on the z axis, your answer is consistent with the exact field (Ex. 5.6), when z ≫ R

$\vec{a} = a\hat{k}$ and $\vec{m} = m\hat{k}$



a) $\vec{m} = I\vec{a} = I(\pi R^2)\hat{k}$

b) $$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

$$= \frac{\mu_0 I \pi R^2}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

c) on the z-axis; $\theta = 0$, $r = z$ and $\hat{r} = \hat{k}$ (for $z>0$), so

$$\vec{B}_{dip} = \frac{\mu_0 I R^2}{4 z^3}(+2\hat{k}) = \frac{\mu_0 I R^2}{2z^3}\hat{k}$$

which is consistent with the result of Ex 5.6 when $z \gg R$

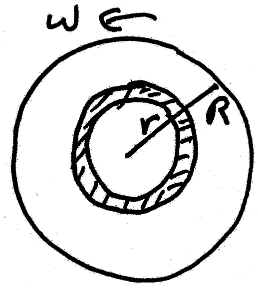
$$\vec{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{k}$$

when $z \gg R$

$$= \frac{\mu_0 I}{2}\frac{R^2}{z^3}\hat{k} = \frac{\mu_0 I R^2}{2z^3}\hat{k}$$ ✓

Problem 5.37

(a) A phonograph record of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω. Find its magnetic dipole moment. **(b)** Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points r > R the potential is that of a perfect dipole.



a) divide the disc into small rings

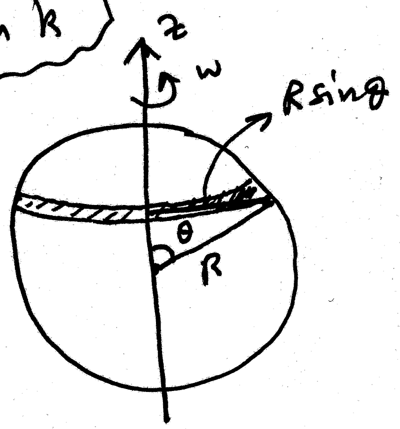
- for a ring of radius r, we have

$$dm = dI(\pi r^2) \;;\; \left\{ dI = \frac{dq}{T} \text{ calculated over one revolution} = \frac{\sigma\, da}{\frac{2\pi}{\omega}} = \frac{\sigma\, 2\pi r\, dr}{\frac{2\pi}{\omega}} = \sigma\omega r\, dr \right.$$

$$= \sigma\omega\pi r^3 dr$$

$$\Rightarrow m = \int_0^R dm$$

$$= \int_0^R \sigma\omega\pi r^3 dr = \frac{1}{4}\pi\sigma\omega R^4 \;;\; \vec{m} = m\hat{k}$$

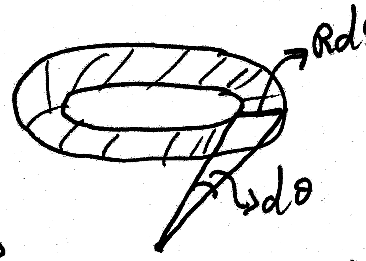


b)
$$dm = dI\,(\text{area}) = dI\left(\pi (R\sin\theta)^2\right)$$

$$= dI\, \pi R^2 \sin^2\theta = \frac{dq}{T}\pi R^2\sin^2\theta$$

$$= \frac{\sigma\, da}{\frac{2\pi}{\omega}}\pi R^2 \sin^2\theta$$

$$= \frac{\sigma\, 2\pi R^2 \sin\theta\, d\theta}{\frac{2\pi}{\omega}}\pi R^2\sin^2\theta$$



$$\left\{ \begin{aligned} da &= 2\pi (R\sin\theta) R\, d\theta \\ &= 2\pi R^2 \sin\theta\, d\theta \end{aligned} \right.$$

$$= \sigma\omega\pi R^4 \sin^3\theta\, d\theta$$

$$\Rightarrow m = \int dm = \sigma\omega\pi R^4 \int_0^\pi \sin^3\theta\, d\theta = \frac{4\pi}{3}\sigma\omega R^4$$

and $\vec{m} = m\hat{k} = m(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$$\Rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi r^2} m \sin\theta\, \overset{(-\hat{\theta}\times\hat{r})}{\hat{\phi}} = \frac{\mu_0 \sigma\omega R^4}{3}\frac{\sin\theta}{r^2}\hat{\phi}$$

which is identical to the field produced by a dipole placed at the origin with an effective $m = \frac{4}{3}\pi\sigma\omega R^4$.