Electromagnetic theory (1)

HW # 8: solutions

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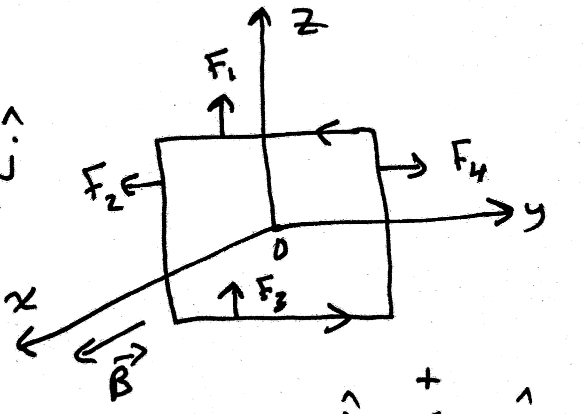
Problem 5.4 Suppose that the magnetic field in some region has the form $\vec{B} = kz\,\hat{i}$ (where k is a constant). Find the force on a square loop (side *a*), lying in the yz-plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.

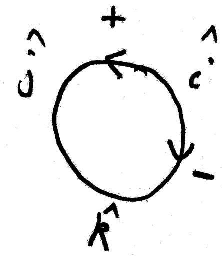
$\vec F = I\vec L\times\vec B$; for straight segment



(2) $\vec F_2 = I\vec L_2\times\vec B = I(-a\hat k)\times(B\hat i) = -IaB\hat j$

(4) $\vec F_4 = I\vec L_4\times\vec B = I(a\hat k)\times(B\hat i) = +IaB\hat j$

so $\vec F_4$ cancels $\vec F_2$



(1) $\vec F_1 = I\vec L_1\times\vec B = I(-a\hat j)\times(B\hat i)$

$= IaB\hat k = Iakz\hat k \xrightarrow{z=a/2} \frac{Ia^2k}{2}\hat k$

(3) $\vec F_3 = I\vec L_3\times\vec B = I(a\hat j)\times(B\hat i) =$

$= -IaB\hat k = -Iakz\hat k \xrightarrow{z=-a/2} \frac{Ia^2k}{2}\hat k$

$\Rightarrow \vec F = \vec F_1 + \vec F_3 = \frac{1}{2}Ika^2\hat k + \frac{1}{2}Ika^2\hat k$

$= Ika^2\hat k$

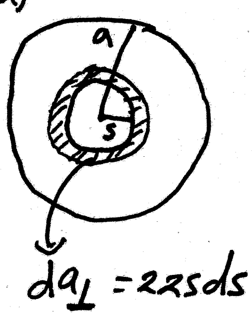
Problem 5.5 A current I flows down a wire of radius *a*. (a) If it is uniformly distributed over the surface, what is the surface current density **K**? (b) If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is J(s)?

a) $\vec K = \frac{d\vec I}{dl_\perp}$; current per unit width. here the length ⊥ to the current is the circumference ($2\pi a$), so

$\vec I = \int d\vec I = \int \vec K\,dl_\perp = \vec K\int dl_\perp = \vec K(2\pi a)$

$= \frac{\vec I}{2\pi a}$

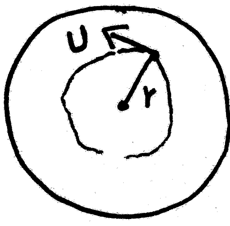
b) $J = \frac{k}{s} \Rightarrow I = \int J\,da_\perp = \int_0^a \frac{k}{s}2\pi s\,ds = 2\pi k a$



$\Rightarrow k = \frac{I}{2\pi a}$ $\Rightarrow J = \frac{k}{s} = \frac{I}{2\pi a s}$

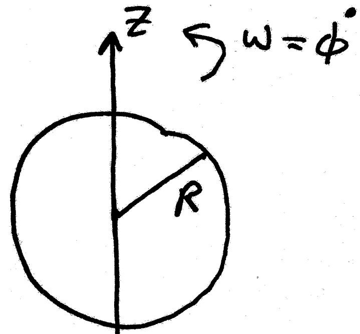
Problem 5.6 **(a)** A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω, what is the surface current density K at a distance r from the center? **(b)** A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density J at any point (r, θ, ϕ) within the sphere.

a)



$u = \omega r$

disc



b)

Solid sphere

$$K = \sigma u = \sigma \omega r \quad \Rightarrow \quad K = K(r)$$

b) Recall that in spherical coordinates

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\text{so } \vec{u} = \frac{dl}{dt} = \underbrace{\dot{r}\,\hat{r}}_{\text{zero}} + \underbrace{r\dot{\theta}\,\hat{\theta}}_{\text{zero}} + r\sin\theta\,\dot{\phi}\,\hat{\phi} \quad ; \quad \dot{\phi} = \omega$$

$$\vec{u} = u_\phi\,\hat{\phi} = r\sin\theta\,\dot{\phi}\,\hat{\phi} = r\sin\theta\,\omega\,\hat{\phi} = r\omega\sin\theta\,\hat{\phi}$$

$$\Rightarrow \vec{J} = \rho\vec{u} = \rho r\omega\sin\theta\,\hat{\phi} = \frac{Q}{\frac{4}{3}\pi R^3}\,r\omega\sin\theta\,\hat{\phi}$$

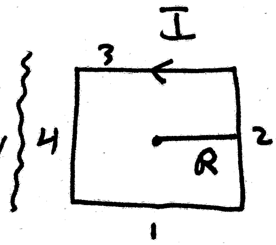
$$= \frac{3Q\omega}{4\pi R^3}\,r\sin\theta\,\hat{\phi}$$

Notice that

$$J = J(r, \theta)$$

Problem 5.8 (a) Find the magnetic field at the center of a square loop, which carries a steady current I. Let R be the distance from center to side. (b) Find the field at the center of a regular n-sided polygon, carrying a steady current I. Again, let R be the distance from the center to any side. (c) Check that your formula reduces to the field at the center of a circular loop, in the limit n → ∞.

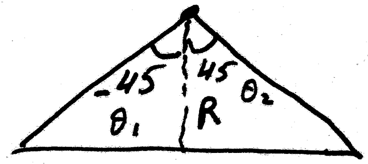
a) the field at the center due to all segments points out of page



$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4$; with $|\vec{B}_1| = |\vec{B}_2| = |\vec{B}_3| = |\vec{B}_4|$

$\Rightarrow \vec{B} = 4\vec{B}_1$, when

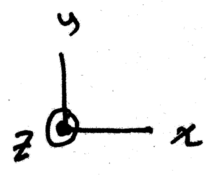
$$B_1 = \frac{\mu_0 I}{4\pi R}(\sin\theta_2 - \sin\theta_1) \; ; \; \theta_2 = -\theta_1 = 45^\circ$$



$$= \frac{\mu_0 I}{4\pi R}(2\sin\theta_2) = \frac{\mu_0 I}{4\pi R} \cdot 2 \cdot \frac{1}{\sqrt{2}}$$

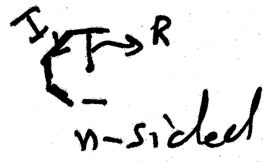
four sides ← $\frac{\pi}{4}$

$$= \frac{\mu_0 I}{2\pi R}\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\mu_0 I \sqrt{2}}{4\pi R}$$



$$\Rightarrow \vec{B} = 4\vec{B}_1 = \sqrt{2}\,\frac{\mu_0 I}{\pi R}\,\hat{k}$$

b) similarly $\vec{B} = n\vec{B}_1 = n\frac{\mu_0 I}{4\pi R}\,2\sin\theta_1$



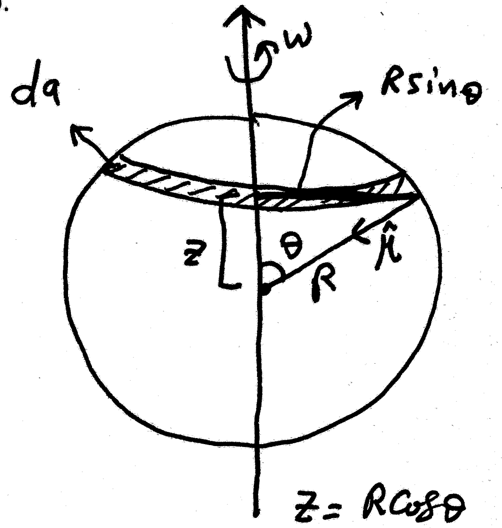
where $\theta_2 = -\theta_1 = \frac{\pi}{n} \Rightarrow \vec{B} = \frac{n\mu_0 I}{4\pi R}\,2\sin\left(\frac{\pi}{n}\right)$

$$= n\frac{\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{n}\right)\hat{k} \text{ (out of page)}$$

c) when $n \to \infty$, $\theta \to$ very small $\Rightarrow \sin\theta \approx \theta \approx \frac{\pi}{n}$

$\Rightarrow \vec{B} = \cancel{n}\frac{\mu_0 I}{2\pi R}\frac{\pi}{\cancel{n}} = \frac{\mu_0 I}{2R}\hat{k}$ which is the same as the field at the center of a circle (out of page)

Problem 5.12 Use the result of Ex. 5.6 to calculate the magnetic field at the center of a uniformly charged spherical shell, of radius R and total charge Q, spinning at constant angular velocity ω.



$z = R\cos\theta$

$da = R^2 \sin\theta \, d\theta \, d\phi$

$$B(0) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \, da$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = 0 - R\hat{r} = -R\hat{r}$$

$$= -R(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$$

and $\vec{K} = \sigma \vec{v} = \sigma(R\sin\theta\,\dot{\phi}\,\hat{\phi})$

$$= \sigma R \sin\theta \, \omega \, \hat{\phi}$$

$$= \sigma R\omega \sin\theta(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

so $\vec{K} \times \vec{\mathscr{r}} = -\sigma R^2 \omega \sin\theta \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\sin\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix} = -\sigma R^2\omega\sin\theta\,(\cos\theta\cos\phi\,\hat{i} + \sin\phi\cos\theta\,\hat{j} + (-\sin\theta\sin^2\phi - \sin\theta\cos^2\phi)\,\hat{k})$

$$= -\sigma R^2 \omega \sin\theta\,[\cos\theta\cos\phi\,\hat{i} + \sin\phi\cos\theta\,\hat{j} - \sin\theta\,\hat{k}]$$

$$\therefore \vec{B}(0) = \frac{\mu_0}{4\pi}(-\sigma R^2 \omega) \int \sin\theta\,[\cos\theta\cos\phi\,\hat{i} + \sin\phi\cos\theta\,\hat{j} - \sin\theta\,\hat{k}] * \frac{1}{\mathscr{r}^3} * R^2 \sin\theta \, d\theta \, d\phi$$

Notice that the first and second terms are zeros as they involve $\int_0^{2\pi} \sin\phi \, d\phi = \int_0^{2\pi} \cos\phi \, d\phi = 0$, so we end up with

$$B(0) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\sigma R^2 \omega \sin^2\theta}{R^3} R^2 \sin\theta \, d\theta \, d\phi = \frac{\mu_0 \sigma}{4\pi} R\omega(2\pi) \int_0^{\pi} \underbrace{\sin^3\theta}_{(1-\cos^2\theta)\sin\theta} \, d\theta$$

$$\underbrace{(1-\cos^2\theta)\sin\theta}_{\frac{4}{3}}$$

$$\boxed{\vec{B}(0) = \frac{\mu_0 \sigma R \omega}{2} \frac{4}{3} = \frac{2}{3} \mu_0 \omega \sigma R \, \hat{k}}$$

but $\sigma = \frac{Q}{4\pi R^2} \Rightarrow \vec{B}(0) = \frac{\mu_0 \omega Q}{6\pi R} \hat{k}$ field at the center of the sphere.

$n = \frac{N}{L}$; N: total # of turns

Example 5.9. Find the magnetic field of a very long solenoid, consisting of *n* closely wound turns per unit length on a cylinder of radius *R*, each carrying a steady current *I*

No field outside as $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

but $I_{enc} = 0 \Rightarrow B = 0$ outside.

Now for the Amperian loop # 2, we have $\vec{B}$ points along the z-direction, so

$$\oint_2 \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

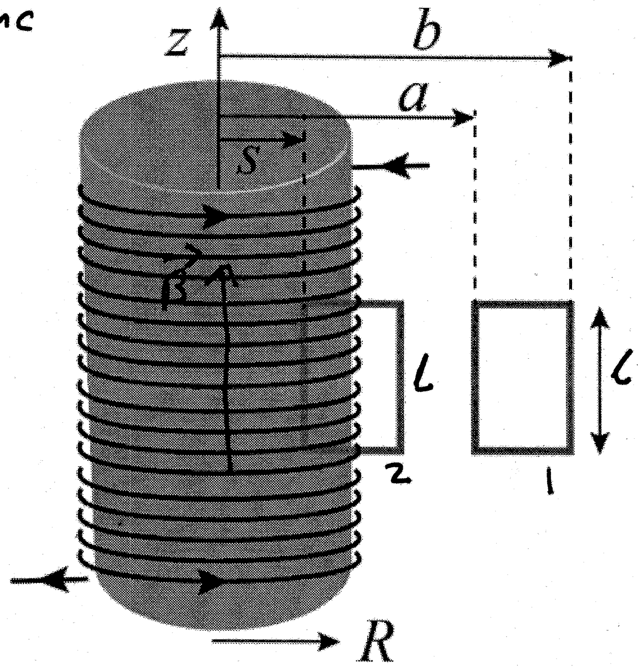
$$B(s)l - \cancel{B(\text{outside})}^{0} l = \mu_0 n I l$$

$$B(s)\cancel{l} = \mu_0 n I \cancel{l}$$

$$B(s) = \mu_0 n I \equiv \text{uniform}$$

so $\vec{B} = \mu_0 n I \hat{k}$; $s<R$

$= 0$; $s>R$



Problem 5.16 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in figure. The inner solenoid (radius *a*) has n_1 turns per unit length, and the outer one (radius *b*) has n_2. Find B in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

i) $\vec{B}_{out} = \mu_0 n_2 I \hat{k}$; $\vec{B}_{inner} = -\mu_0 n_1 I$

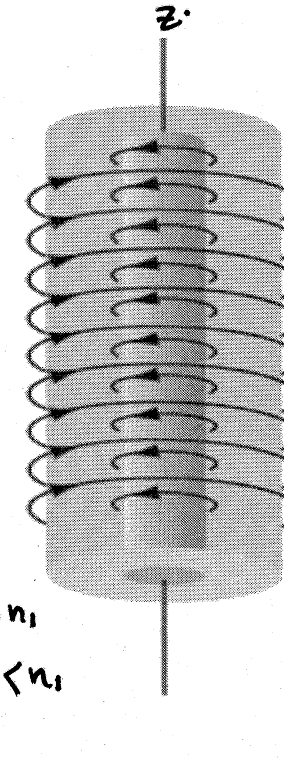
$$\vec{B} = \mu_0 I (n_2 - n_1) \hat{k}$$

ii) the field is due to the outer solenoid only $\vec{B} = \mu_0 n_2 I \hat{k}$

iii) $\vec{B}_{out} = 0$ and $\vec{B}_{in} = 0$ outside

$\Rightarrow \vec{B} = 0$ outside

Notice that in (i) the direction of the net field depends on $(n_2 - n_1) = \begin{cases} \text{up} ; n_2 > n_1 \\ \text{down} ; n_2 < n_1 \end{cases}$



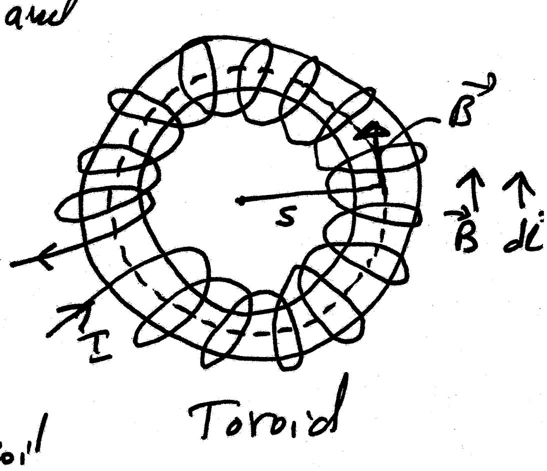
for Toroid, $\vec{B}$ is tangential inside and is given by

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

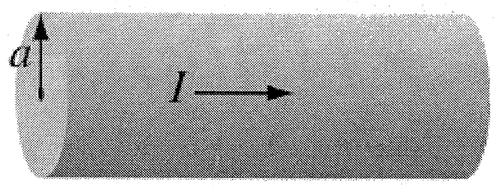
$$B(2\pi s) = \mu_0 N I$$

$$\Rightarrow \vec{B} = \frac{\mu_0 N I}{2\pi s} \hat{\phi} \text{ ; inside the coil}$$

$$= 0 \text{ ; outside the coil}$$



Problem 5.14 A steady current I flows down a long cylindrical wire of radius *a*. Find the magnetic field, both inside and outside the wire, if (a) The current is uniformly distributed over the outside surface of the wire. (b) The current is distributed in such a way that **J** is proportional to s, the distance from the axis.



a) here No current inside, only current exists on the surface, so the field is zero inside and

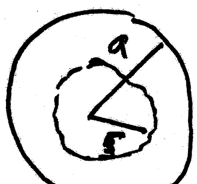
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B(2\pi s) = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

outside $(s>a)$

$$\therefore \vec{B} = \begin{cases} 0 & , s<a \\ \frac{\mu_0 I}{2\pi s} \hat{\phi} ; & s>a \end{cases}$$

b) $J = ks$; outside $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$

Now inside $\oint \vec{B} \cdot d\vec{l} = \mu_0 I'$; what is I'



the total current inside (for $s \leq a$) is

$$I = \int_0^a J \, da = \int_0^a ks(2\pi s) ds = \frac{2\pi k a^3}{3} \Rightarrow k = \frac{3I}{2\pi a^3}$$

inside

$$\text{so } J = ks = \frac{3I}{2\pi a^3} s \text{ , } \Rightarrow I' = \int_0^s J \, da = \int_0^s ks(2\pi s) ds = \frac{2\pi k s^3}{3}$$

$$= \frac{2\pi}{3} k s^3 = \frac{2\pi}{3}\left(\frac{3I}{2\pi a^3}\right) s^3 = \frac{I s^3}{a^3}$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I'$$

$$B(2\pi s) = \frac{\mu_0 I s^3}{a^3}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I s^2}{2\pi a^3} \hat{\phi} \text{ ; So; } \vec{B} = \begin{cases} \frac{\mu_0 I s^2}{2\pi a^3} \hat{\phi} ; & s<a \\ \frac{\mu_0 I}{2\pi s} \hat{\phi} ; & s>a \end{cases}$$