Electromagnetic theory (1)

HW # 7 - Solution

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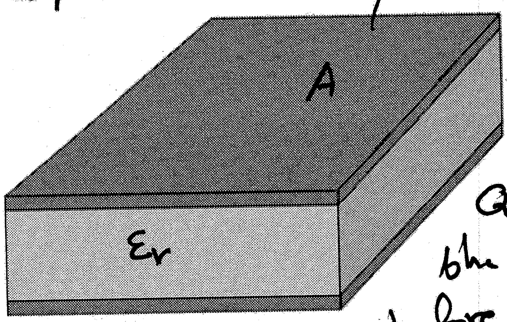
Example 4.6. A parallel-plate capacitor is filled with insulating material of dielectric constant ϵ_r. What effect does this have on its capacitance?

without dielectric the vacuum capacitance is $C_{vac} = \frac{\varepsilon_0 A}{d}$; $\Delta V_{vac} = E_{vac} d$

$\Delta V = \frac{\Delta V_{vac}}{\varepsilon_r}$; $\vec{E} = \frac{\vec{E}_{vac}}{\varepsilon_r}$



Q stays the same before and after

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{\Delta V_{vac}}{\varepsilon_r}} = \varepsilon_r \frac{Q}{\Delta V_{vac}} = \varepsilon_r C_{vac}$$

also $U = \frac{U_{vac}}{\varepsilon_r}$

Problem 4.18 The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate −σ.

(a) Find the electric displacement D in each slab. **(b)** Find the electric field E in each slab.

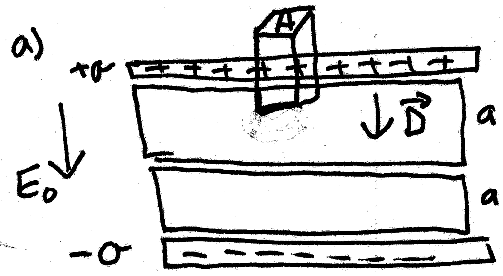
$\varepsilon_r^1 = 2$

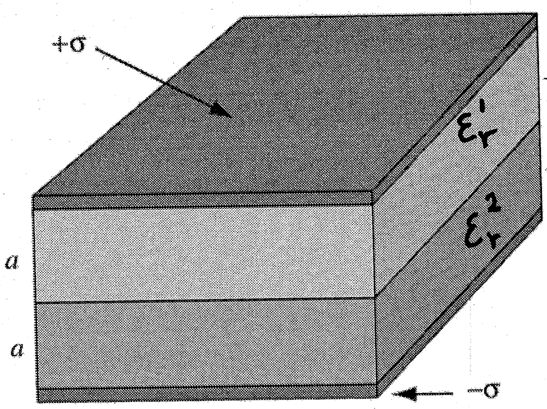
(c) Find the polarization P in each slab. **(d)** Find the potential difference between the plates.

$\varepsilon_r^2 = 1.5 = 3/2$

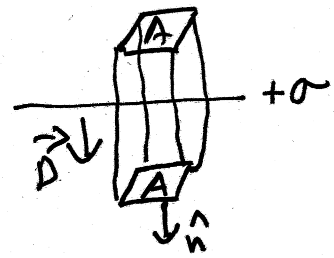
(e) Find the location and amount of all bound charge. **(f)** Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

a)





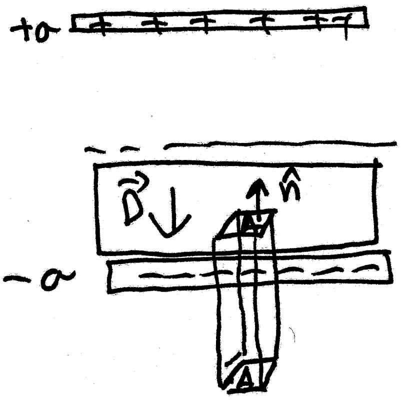
$\oint \vec{D} \cdot d\vec{a} = Q_f$, take the gaussian surface as a Pillbox as shown. Notice that there is no field outside the plates, so there will be no flux through the top surface of the pill box. both $\vec{D}$ and $\hat{n}$ point downward, so



$D A = \sigma A \Rightarrow D = \sigma$ and points downward

similarly on the bottom plate, we have

Notice here that $\vec{D}$ and $\hat{n}$ are opposite ($\theta = 180°$), so



$$\oint \vec{D} \cdot d\vec{a} = Q_f$$

$$-DA = -\sigma A \Rightarrow D = \sigma$$

and points downward

b) $\vec{D} = \varepsilon \vec{E}$

$$\vec{E}_1 = \frac{\vec{D}_1}{\varepsilon_1} = \frac{\vec{D}_1}{\varepsilon_0 \varepsilon_r^1} = \frac{\vec{D}_1}{2\varepsilon_0} = \frac{\sigma}{2\varepsilon_0} \text{ downward}$$

similarly $\vec{E}_2 = \frac{\vec{D}_2}{\varepsilon_2} = \frac{\vec{D}_2}{\varepsilon_0 \varepsilon_r^2} = \frac{\sigma}{\varepsilon_0 \frac{3}{2}} = \frac{2\sigma}{3\varepsilon_0}$ downward

so in general $\vec{E} = \frac{\sigma}{\varepsilon_r \varepsilon_0}$ downward

c) $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \chi_e \frac{\sigma}{\varepsilon_r \varepsilon_0} = \chi_e \frac{\sigma}{\varepsilon_r}$; but $\varepsilon_r = 1 + \chi_e$
$\Rightarrow \chi_e = \varepsilon_r - 1$

$= \frac{\varepsilon_r - 1}{\varepsilon_r} \sigma = \left(1 - \frac{1}{\varepsilon_r}\right) \sigma$

$\vec{P}$ points downward as $\vec{E}$

$$= \begin{cases} P_1 = \left(1 - \frac{1}{2}\right)\sigma = \frac{\sigma}{2} \\ P_2 = \left(1 - \frac{2}{3}\right)\sigma = \frac{\sigma}{3} \end{cases}$$

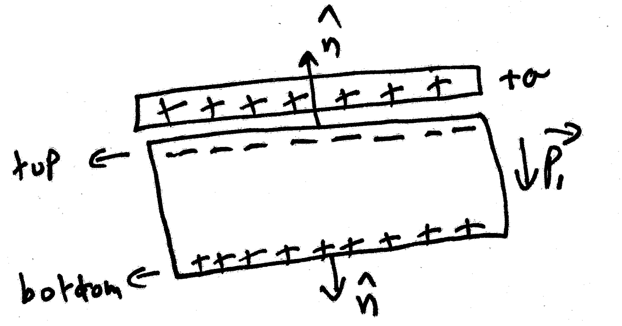
d) $\Delta V = \Delta V_1 + \Delta V_2 = E_1 a + E_2 a = \frac{\sigma}{2\varepsilon_0} a + \frac{2\sigma}{3\varepsilon_0} a = \frac{\sigma a}{\varepsilon_0}\left(\frac{1}{2} + \frac{2}{3}\right) = \frac{7}{6} \frac{\sigma a}{\varepsilon_0}$

e) There are no bound volume charges (constant polarization) $\rho_b^1 = -\nabla \cdot \vec{P}_1 = 0$ and $\rho_b^2 = -\nabla \cdot \vec{P}_2 = 0$ as $P_1 =$ constant and $P_2 =$ constant

the bound surface charge density $\sigma_b = \vec{P} \cdot \hat{n}$, so for slab 1

$$\sigma^1_{b(top)} = \vec{P}_1 \cdot \hat{n} = -P_1 = -\frac{\sigma}{2}$$

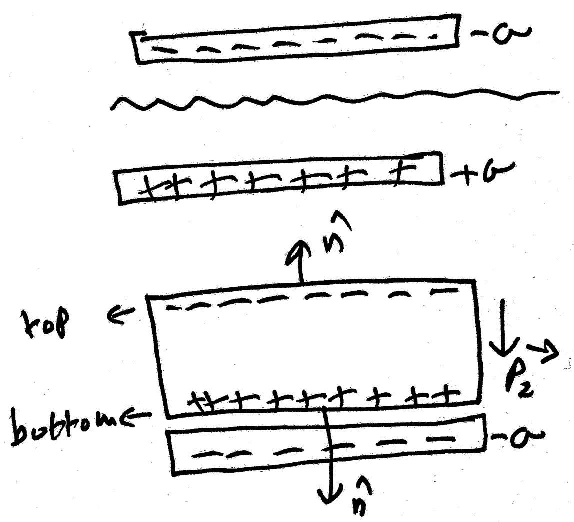
$$\sigma^1_{b(bot)} = \vec{P}_1 \cdot \hat{n} = P_1 = +\frac{\sigma}{2}$$



- for slab 2, we have

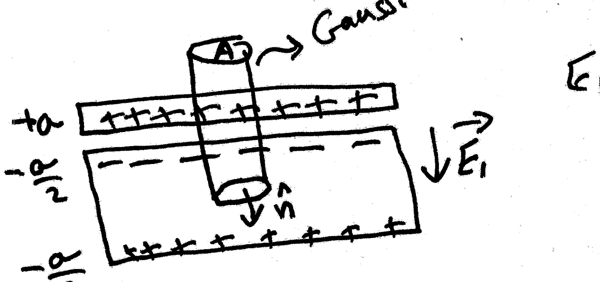
$$\sigma^2_{b(top)} = \vec{P}_2 \cdot \hat{n} = -P_2 = -\frac{\sigma}{3}$$

$$\sigma^2_{b(bot)} = \vec{P}_2 \cdot \hat{n} = +P_2 = +\frac{\sigma}{3}$$



e) for slab 1, using Gauss's law

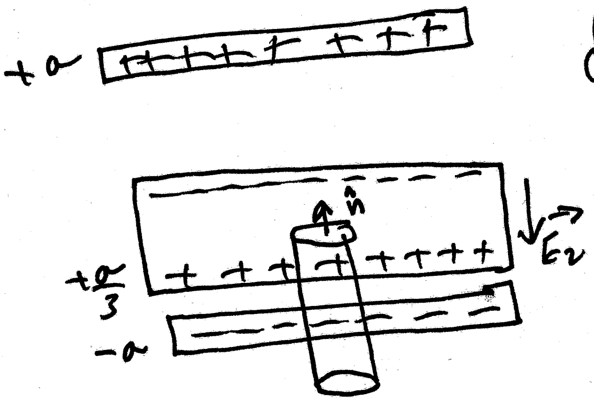
$$\oint \vec{E}_1 \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$



$$E_1 A = \frac{\sigma A - \frac{\sigma}{2} A}{\varepsilon_0} \Rightarrow E_1 = \frac{\sigma}{2\varepsilon_0}$$

downward as expected from part (b)

similarly for slab 2, we have



$$\oint \vec{E}_2 \cdot da = \frac{Q_{enc}}{\varepsilon_0}$$

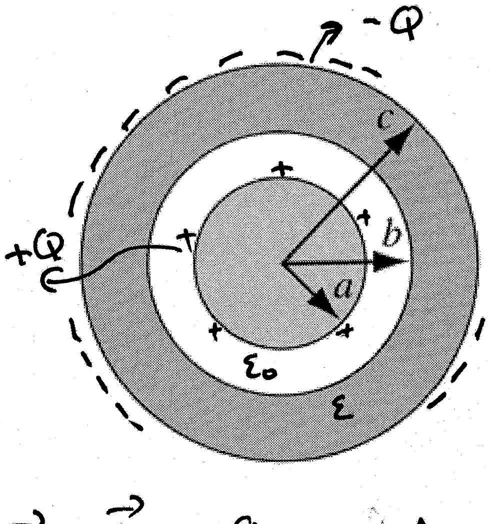
$$-E_2 \cancel{A} = \frac{\left(\frac{\sigma}{3} - \sigma\right)\cancel{A}}{\varepsilon_0}$$

$$-E_2 = -\frac{2}{3}\frac{\sigma}{\varepsilon_0} \Rightarrow E_2 = \frac{2\sigma}{3\varepsilon_0}$$

downward.

as expected from (b).

Problem 4.21 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c. The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable.



- for $a < s < b$ and applying Gauss's law, we get

$$\oint \vec{D} \cdot d\vec{a} = Q_{f\,enc}$$

taking a cylindrical Gaussian surface of radius r and length l, we get

$$D(2\pi s l) = Q \Rightarrow \vec{D} = \frac{Q}{2\pi s l}\hat{s} \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{Q}{2\pi\varepsilon_0 s l}\hat{s}$$

- for $b < r < c$, similarly and taking a cylindrical gaussian surface between b and c

$$\oint \vec{D} \cdot d\vec{a} = Q_{f\,enc}$$

$$D(2\pi s l) = Q \Rightarrow \vec{D} = \frac{Q}{2\pi s l}\hat{s}; \quad \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{2\pi\varepsilon s l}\hat{s}$$

$$\text{so } \Delta V = -\int_a^c \vec{E} \cdot d\vec{l} = -\int_a^b \frac{Q}{2\pi\varepsilon_0 s l}\,ds - \int_b^c \frac{Q}{2\pi\varepsilon s l}; \quad \varepsilon_r = \frac{\varepsilon}{\varepsilon_0} \Rightarrow \varepsilon = \varepsilon_r\varepsilon_0 \text{ or}$$

$$= -\frac{Q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right) - \frac{Q}{2\pi\varepsilon l}\ln\left(\frac{c}{b}\right) = \frac{-Q}{2\pi\varepsilon_0 l}\left[\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)\right]$$

$$= -\frac{Q}{2\pi\varepsilon_0 l}\left[\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)\right] \overset{\text{or}}{=} -\frac{Q}{2\pi\varepsilon_0 l}\left[\ln\left(\frac{b}{a}\right) + \frac{\varepsilon_0}{\varepsilon}\ln\left(\frac{c}{b}\right)\right]$$

$$\text{so } C = \frac{Q}{|\Delta V|} = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)}$$

$$\text{so } \frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)}$$

QED

Problem 4.26: A spherical conductor, of radius a, carries a charge Q (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration (Eq. 4.58).

from example 4.5, we found

$$\vec{D} = \begin{cases} 0 , & r<a \\ \dfrac{Q}{4\pi r^2}\hat{r} , & r>a \end{cases}$$

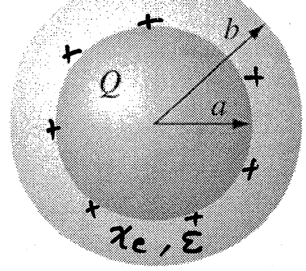


FIGURE 4.29

$d\tau = 4\pi r^2 dr$

$$\vec{E} = \begin{cases} 0 , & r<a \\ \dfrac{Q}{4\pi\varepsilon r^2}\hat{r} , & a<r<b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} , & r>b \end{cases}$$

$$W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau = \frac{4\pi}{2}\left[\int_a^b \frac{Q}{4\pi r^2}\frac{Q}{4\pi\varepsilon r^2} r^2 dr + \int_b^\infty \frac{Q}{4\pi r^2}\frac{Q}{4\pi\varepsilon_0 r^2} r^2 dr\right]$$

$$= \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left[\frac{1}{\varepsilon}\int_a^b \frac{dr}{r^2} + \frac{1}{\varepsilon_0}\int_b^\infty \frac{dr}{r^2}\right]$$

Now $\varepsilon = \varepsilon_0(1+\chi_e)$

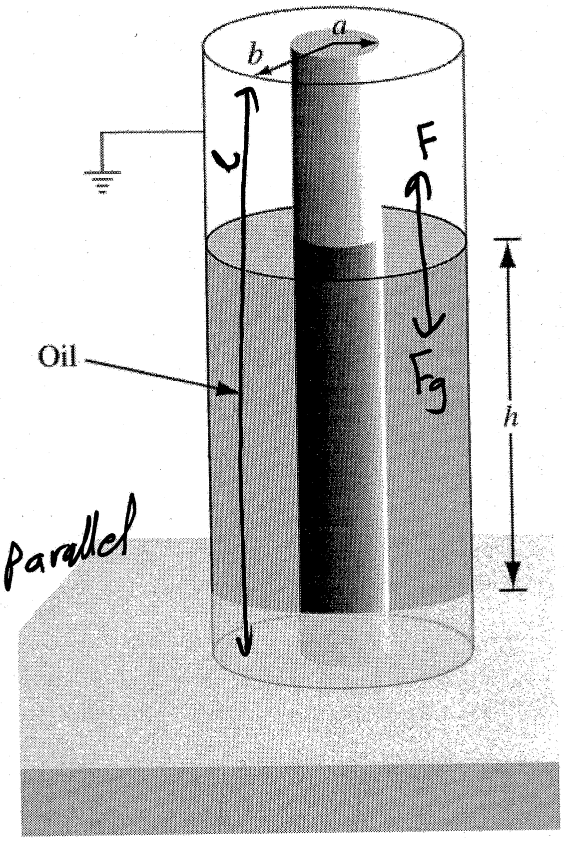
$$= \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a} + \frac{\chi_e}{b}\right]$$

Problem 4.28 Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded. To what height (h) does the oil rise, in the space between the tubes?

at equilibrium $F = F_g$; $F = -\frac{dW}{dh}$

$F_g = mg$



$$\Rightarrow -\frac{dW}{dh} = mg$$

$$= \rho V' g = \rho \pi (b^2 - a^2) h g$$

now $W = \frac{Q^2}{2C}$; what is C?

$$C = C_{vac} + C_{dile} = \frac{4\pi\varepsilon_0 (l-h)}{2\ln(b/a)} + \frac{4\pi\varepsilon h}{2\ln(b/a)}$$; parallel

$$= \frac{2\pi\varepsilon_0}{\ln(b/a)}\left[l - h + \varepsilon_r h\right\} ; \quad \varepsilon = \varepsilon_r \varepsilon_0$$

$$= \frac{2\pi\varepsilon_0}{\ln(b/a)}\left[l + h(\varepsilon_r - 1)\right\} ; \quad \varepsilon_r = 1 + \chi_e$$

$F_g = mg$

$$= \frac{2\pi\varepsilon_0}{\ln(b/a)}\left[l + \chi_e h\right\} \Rightarrow \frac{dC}{dh} = \frac{2\pi\varepsilon_0 \chi_e}{\ln(b/a)}$$

$$\Rightarrow W = \frac{Q^2}{2C} \Rightarrow \frac{dW}{dh} = \frac{1}{2} Q^2 \left(\frac{-1}{C^2}\frac{dC}{dh}\right) = -\frac{1}{2} V^2 \cdot \frac{2\pi\varepsilon_0 \chi_e}{\ln(b/a)}$$

$$\therefore -\frac{dW}{dh} = \rho\pi(b^2 - a^2) g h \Rightarrow \frac{1}{\cancel{2}} V^2 \cdot \frac{\cancel{2}\cancel{\pi}\varepsilon_0 \chi_e}{\ln(b/a)} = \rho\cancel{\pi}(b^2 - a^2) g h$$

height of oil at equilibrium

$$\Rightarrow \boxed{h = \frac{\varepsilon_0 V^2 \chi_e}{\rho (b^2 - a^2) g \ln(b/a)}}$$

QED