Electromagnetic theory (1)

HW # 6 – Solution

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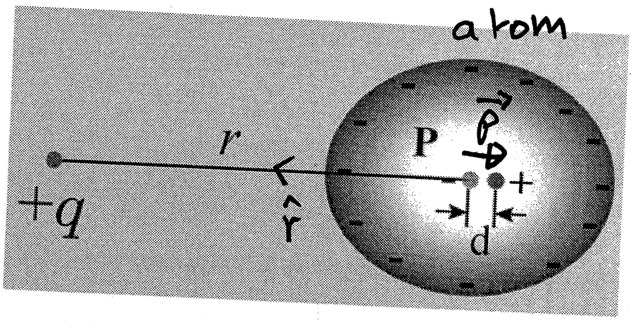
Problem 4.4: A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.

the field of q at the position of $\vec{p}$ is given by



$\vec{E} = k_e \frac{q}{r^2} (-\hat{r})$. This is the external field.

Now at equilibrium, $E_e = E$, so $\vec{p} = \alpha \vec{E}$

$= -k_e \frac{\alpha q}{r^2} \hat{r}$

now the field of the dipole at the location of the charge q is

$\vec{E}_{dip}(r, \theta) = \frac{k_e p}{r^3} (2 \cos\theta \, \hat{r} + \sin\theta \, \hat{\theta})$; with $\theta = \pi$

$= \frac{k_e^2 \alpha q}{r^5} (-2\hat{r})$ to the right

$\Rightarrow \vec{F}_q = q \vec{E}_{dip} = -\frac{2 k_e^2 \alpha q^2 \hat{r}}{r^5}$ to the right

this is the force exerted by dipole on the charge q ; which is an attractive force.

Problem 4.5: In Fig. 4.6, $\mathbf{p}_1$ and $\mathbf{p}_2$ are (perfect) dipoles a distance r apart. What is the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$? What is the torque on $\mathbf{p}_2$ due to $\mathbf{p}_1$? [In each case, I want the torque on the dipole about its own center. If it bothers you that the answers are not equal and opposite, see Prob. 4.29.]

Field of $\vec{P_1}$ at $\vec{P_2}$ is

$$\vec{E_1}(r,\theta) = \frac{k_e P_1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

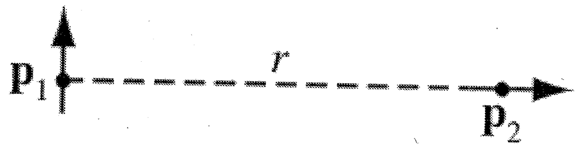


FIGURE 4.6

$\downarrow \vec{E_1}$

here $\theta = \pi/2 \Rightarrow \vec{E_1} = \frac{k_e P_1}{r^3}\hat{\theta}$ Points downward

Torque on $\vec{P_2}$ is

$$\vec{N_2} = \vec{P_2} \times \vec{E_1} = P_2 E_1 \sin(\pi/2) = P_2 E_1 = k_e \frac{P_1 P_2}{r^3}$$ into the page by R. H. Rule

similarly,

Field of $\vec{P_2}$ at $\vec{P_1}$ is

$$\vec{E_2}(r,\theta) = \frac{k_e P_2}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right);\ \text{with } \theta = \pi$$

$$= \frac{k_e P_2}{r^3}(-2\hat{r})$$ points to the right

$\Rightarrow$ Torque on $\vec{P_1}$ is

$$\vec{N_1} = \vec{P_1} \times \vec{E_2} = P_1 E_2 \sin(\pi/2) = P_1 E_2$$

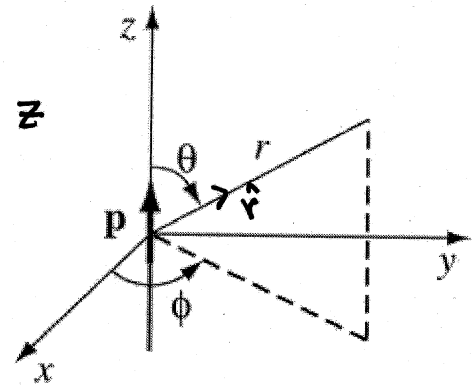
$$= 2\frac{k_e P_1 P_2}{r^3}$$ Points into the page by R. H. Rule

Note that $N_1 \neq N_2$
and both point into the page

Problem 4.8 Show that the interaction energy of two dipoles separated by a displacement **r** is

[Hint: use Eq. 3.104. in textbook]

$$U = k_e \frac{1}{r^3}\left[\vec{P}_1\cdot\vec{P}_2 - 3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r})\right]$$



- we already found that for a point dipole located at the origin and pointed along z direction, $\vec{E}_{dip}(r,\theta) =$

$$\vec{E}_{dip}(r,\theta) = k_e \frac{P}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \quad \text{---(1)}$$

This equation can be written as

$$\vec{E}_{dip}(r,\theta) = k_e \frac{1}{r^3}\left[\underbrace{3(\vec{P}\cdot\hat{r})\hat{r} - \vec{P}}\right] \quad \text{--- (2)} \qquad \text{check?!}$$

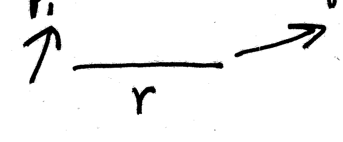
$$\vec{P} = P_r\hat{r} + P_\theta\hat{\theta} = (\vec{P}\cdot\hat{r})\hat{r} + (\vec{P}\cdot\hat{\theta})\hat{\theta} \quad ; \quad \text{with } \vec{P} = P\hat{k}$$

where in spherical coordinates

$$\left.\begin{aligned}\hat{r} &= \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\\ \hat{\theta} &= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\end{aligned}\right\} \Rightarrow \begin{aligned}\vec{P}\cdot\hat{r} &= P\cos\theta\\ \vec{P}\cdot\hat{\theta} &= -P\sin\theta\end{aligned}$$

$$\begin{aligned}\text{so } 3(\vec{P}\cdot\hat{r})\hat{r} - \vec{P} &= 3P\cos\theta\,\hat{r} - \vec{P}\\ &= 3P\cos\theta\,\hat{r} - (P\cos\theta\,\hat{r} - P\sin\theta\,\hat{\theta})\\ &= 2P\cos\theta\,\hat{r} + P\sin\theta\,\hat{\theta}\\ &= P(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\end{aligned}$$

So equations (1) and (2) are the same ✓ [See problem 3.36]

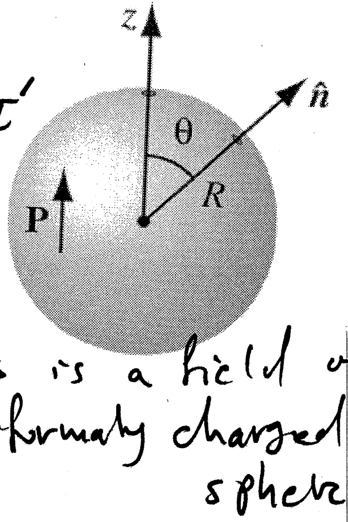


$$\begin{aligned}U = -\vec{P}_1\cdot\vec{E}_2 &= -\vec{P}_1\cdot\frac{k_e}{r^3}\left(3(\vec{P}_2\cdot\hat{r})\hat{r} - \vec{P}_2\right)\\ &= -\frac{k_e}{r^3}\left(3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r}) - \vec{P}_1\cdot\vec{P}_2\right)\\ &= \frac{k_e}{r^3}\left[\vec{P}_1\cdot\vec{P}_2 - 3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r})\right]\end{aligned}$$

QED

Problem 4.12: Calculate the potential of a uniformly polarized sphere (Ex. 4.2) directly from Eq. 4.9

$\vec{P} = P\hat{k}$; using eqⁿ 4.9



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{P(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d\tau'$$

but $\vec{P}$ is constant

$$= \vec{P}\cdot\left[\frac{1}{4\pi\varepsilon_0}\int_V \frac{(\vec{r}-\vec{r}\,')\,d\tau'}{|\vec{r}-\vec{r}\,'|^3}\right]$$

This is a field of uniformaly charged sphere

Now using $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')\,d\tau'}{|\vec{r}-\vec{r}\,'|^3}$

for uniformly charged sphere $\vec{E} = \frac{1}{4\pi\varepsilon_0}\rho\int \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\,d\tau'$

$$\Rightarrow \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\,d\tau' = \frac{\vec{E}}{\rho};$$

$$\Rightarrow V(\vec{r}) = \vec{P}\cdot\frac{\vec{E}}{\rho}\ ;\ \text{but } \vec{E} = \begin{cases} k_e\dfrac{Q}{r^2}\hat{r}\ ; & r>R \\ k_e\dfrac{Q}{R^3}r\,\hat{r}, & r<R \end{cases}$$

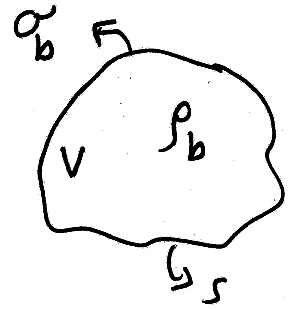
$$\Rightarrow \vec{E} = \begin{cases} k_e\dfrac{\rho\frac{4}{3}\pi R^3}{r^2}\hat{r}, & r>R \\ k_e\,\rho\dfrac{\frac{4}{3}\pi r^3}{r^2}\hat{r}, & r<R \end{cases} = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0}\dfrac{1}{r^2}\hat{r}, & r>R \\ \dfrac{\rho}{3\varepsilon_0}r\,\hat{r}, & r<R \end{cases}$$

$$\Rightarrow V(r) = \vec{P}\cdot\frac{\vec{E}}{\rho} = \begin{cases} \dfrac{R^3}{3\varepsilon_0 r^2}\vec{P}\cdot\hat{r} = \dfrac{R^3 P\cos\theta}{3\varepsilon_0 r^2}, & r>R \\ \dfrac{\vec{P}\cdot(r\hat{r})}{3\varepsilon_0} = \dfrac{r}{3\varepsilon_0}\vec{P}\cdot\hat{r} = \dfrac{Pr\cos\theta}{3\varepsilon_0}, & r<R \end{cases}$$

where $\vec{P}\cdot\hat{r} = P(\hat{k}\cdot\hat{r}) = P\cos\theta$

Problem 4.14: When you polarize a neutral dielectric, the charge moves a bit, but the total remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

Total charge is



$$Q = \oint_S \sigma_b \, da' + \int_V \rho_b \, d\tau'$$

$$= \oint_S \vec{P}(\vec{r}') \cdot \hat{n} \, da' - \int_V \nabla' \cdot \vec{P}(\vec{r}') \, d\tau'$$

$$= \oint_S \vec{P}(\vec{r}') \cdot d\vec{a}' - \int_V \nabla' \cdot \vec{P}(\vec{r}') \, d\tau'$$

$\downarrow$ using divergence theorem

$$= \int_V \nabla' \cdot \vec{P}(\vec{r}') \, d\tau' - \int_V \nabla' \cdot \vec{P}(\vec{r}') \, d\tau'$$

$$= \text{zero}$$