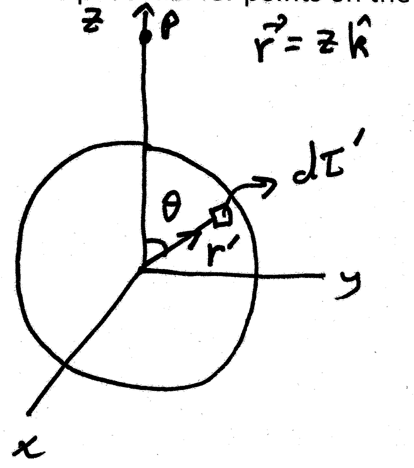
Electromagnetic theory (1)

HW #5 - Solution

Dr. Gassem Alzoubi

Problem 3.27 A sphere of radius R, centered at the origin, carries charge density $\rho(r,\theta) = k\frac{R}{r^2}(R-2r)sin\theta$, where k is a constant, and r, Θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.



$$V_P = V_{mono} + V_{dip} + V_{quad}$$

i) monopole $V_{mono} = k_e \frac{Q}{r}$

but $Q = \int \rho(\vec{r}')d\tau' =$

$$= \int k\frac{R}{\cancel{r'^2}}(R-2r')\sin\theta \, \cancel{r'^2}\sin\theta \, dr' d\theta \, d\phi$$

$$= kR\underbrace{\int_0^R (R-2r')dr'}_{=zero}\int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta \, d\theta = 0$$

$\Rightarrow V_{mono} = zero$

ii) dipole $V_{dip} = \frac{k_e}{r^2}\int r'\cos\theta \, \rho(\vec{r}')d\tau'$

$$V_{dip} = \frac{k_e}{r^2}\left[\int r'\cos\theta \, k\frac{R}{\cancel{r'^2}}(R-2r')\sin\theta \, \cancel{r'^2}\sin\theta \, d\theta \, dr' d\phi\right]$$

$$= \frac{k_e}{r^2}\left[\int_0^R r'(R-2r')dr'\int_0^{2\pi} d\phi \underbrace{\int_0^{\pi}\sin^2\theta\cos\theta \, d\theta}_{=zero}\right] = 0$$

iii) $V_{quad} = \frac{k_e}{r^3}\int r'^2 \frac{1}{2}(3\cos^2\theta - 1)\rho(\vec{r}')d\tau'$

$$= \frac{k_e}{r^3}\int \cancel{r'^2}\frac{1}{2}(3\cos^2\theta - 1)k\frac{R}{\cancel{r'^2}}(R-2r')\sin\theta \, r'^2\sin\theta \, d\theta \, dr' d\phi$$

$$= k_e\frac{kR}{2r^3}\left[\underbrace{\int_0^R r'^2(R-2r')dr'}_{-\frac{R^4}{6}}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\int_0^{\pi}(3\cos^2\theta - 1)\sin^2\theta \, d\theta\right]$$

the θ-integral $\int_0^{\pi} (3\cos^2\theta - 1)\sin^2\theta$; using $\cos^2\theta = 1 - \sin^2\theta$

$3\cos^2\theta = 3 - 3\sin^2\theta$

$$= \int_0^{\pi} (3 - 3\sin^2\theta - 1)\sin^2\theta \, d\theta$$

$$= \int_0^{\pi} (2 - 3\sin^2\theta) \quad \sin^2\theta \, d\theta$$

$$= 2\int_0^{\pi} \sin^2\theta \, d\theta \; - 3\int_0^{\pi} \sin^4\theta \, d\theta = 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right)$$

$$= -\frac{\pi}{8}$$

$$\therefore V_{quad} = \frac{k_e k R}{\cancel{2} r^3}\left(-\frac{R^4}{6}\right)(\cancel{2}\pi)\left(-\frac{\pi}{8}\right) = \frac{k_e \pi^2 R^5 k}{48 r^3}$$

here $r = z$

$$\Rightarrow V_p(z) = \frac{1}{4\pi\varepsilon_0} \, \frac{k\pi^2 R^5}{48 z^3}$$

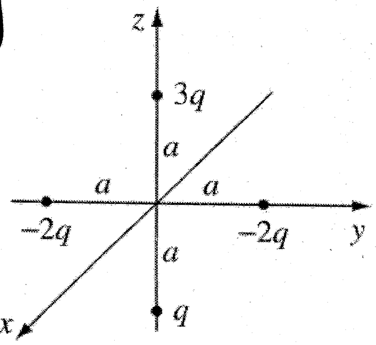
Problem 3.29 Four particles (one of charge q, one of charge 3q, and two of charge −2q) are placed as shown in Fig. 3.31, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)

$Q = 3q + q - 2q - 2q = 0 \Rightarrow$ No monopole potential

$$\vec{p} = \sum q_i \vec{r}_i' = (-2q)a\hat{j} + (3q)a\hat{k} + (-2q)a(-\hat{j}) + qa(-\hat{k})$$

$$= -\cancel{2qa\hat{j}} + 3qa\hat{k} + \cancel{2qa\hat{j}} - qa\hat{k}$$

$$= 2qa\,\hat{k}$$



$$\Rightarrow V_{dip} = k_e \frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{k_e \, 2qa \; \hat{k}\cdot\hat{r}}{r^2} = k_e \frac{2qa}{r^2}\cos\theta$$

recall $\hat{k}\cdot\hat{r} = \hat{k}\cdot(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$

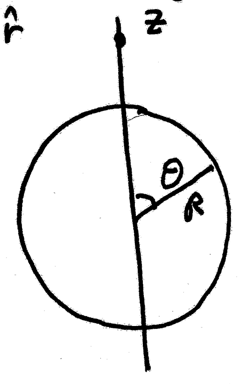
Problem 3.30 In Ex. 3.9, we derived the exact potential for a spherical shell of radius R, which carries a surface charge σ = k cos θ. (a) Calculate the dipole moment of this charge distribution.(b) Find the approximate potential, at points far from the sphere, and compare the exact answer (Eq. 3.87). What can you conclude about the higher multipoles?

$$Q = \int \rho(\vec{r}')\,d\tau' \rightarrow \int \sigma(\vec{r}')\,da' \;;\; d\vec{a}' = r'^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$= \int k\cos\theta\, r'^2 \sin\theta\, d\theta\, d\phi \;;\; r' = R \text{ at surface}$$

$$= kR \int_0^{2\pi} d\phi \int_0^{2\pi} \sin\theta \cos\theta\, d\theta = 2\pi k R \left.\frac{\sin^2\theta}{2}\right|_0^{\pi} = 0$$

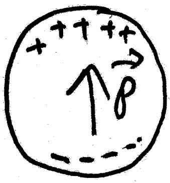
No monopole potential



- for the dipolar potential,

Notice that

$$\sigma(\theta) = \begin{cases} +k, & \theta = 0 \\ -k, & \theta = \pi \end{cases} \Rightarrow$$



$\vec{p}$ is clearly pointing along + z-direction $\vec{p} = p\,\hat{k}$

$$p = \int r'\cos\theta\, \rho(r')\,d\tau' \rightarrow \int r'\cos\theta\, \sigma(r')\,da' \;;\; r' = R$$

$$= \int r'\cos\theta\, k\cos\theta\, r'^2 \sin\theta\, d\theta\, d\phi$$

$$= R^3 k \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = 2\pi k R^3 (-1) \underbrace{\int_0^{\pi} \cos^2\theta\,(-\sin\theta)\,d\theta}_{\left.\frac{\cos^3\theta}{3}\right|_0^{\pi}}$$

$$= -\frac{2\pi k R^3}{3} \cos^3\theta \Big|_0^{\pi} = \frac{-2\pi k R^3}{3}\{-1-1\}$$

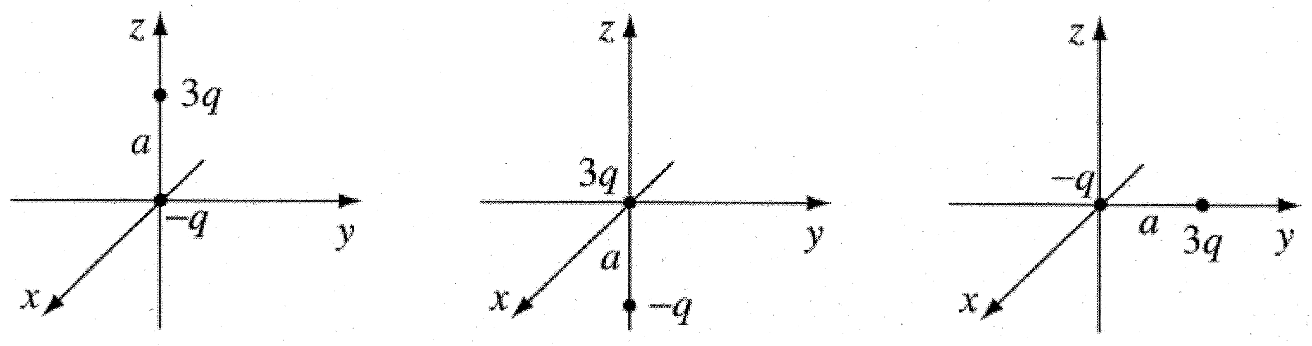
$$= \frac{4\pi k R^3}{3} \Rightarrow \boxed{\vec{p} = \frac{4\pi k R^3}{3}\hat{k}}$$

$$V_{dip} = k_e \frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{\cancel{4\pi} k R^3 (\hat{k}\cdot\hat{r})}{3 r^2} \;;\; \text{but } \hat{k}\cdot\hat{r} = \cos\theta$$

$$= \frac{kR^3}{3\varepsilon_0}\frac{\cos\theta}{r^2}$$ exactly as obtained in ex 3.9.

since this is exact, all other multipoles are zero

Problem 3.32 Two point charges, 3q and −q, are separated by a distance a. For each of the arrangements shown , find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both the monopole and dipole contributions).



(a) (b) (c)

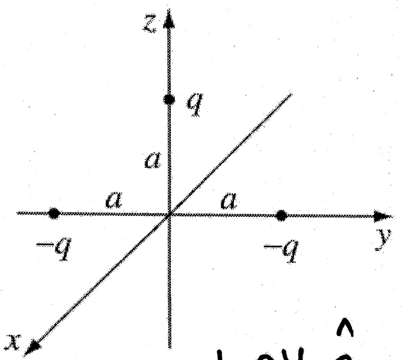
in spherical $\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

(a) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right] = \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]}$

(b) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]}.$

(c) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{y}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$ (from Eq. 1.64

Problem 3.34 Three point charges are located as shown in Fig. 3.38, each a distance a from the origin. Find the approximate electric field at points far from the origin. Express your answer in spherical coordinates, and include the two lowest orders in the multipole expansion.



$Q = -q - q + q = -q \Rightarrow V_{mono} = k_e\frac{(-q)}{r}$

$\vec{P} = \sum_{i=1}^{3} q_i \vec{r_i}' = \cancel{(-q)a\hat{j}} + qa\hat{k} + \cancel{(-q)a(-\hat{j})}$

$= qa\hat{k}$

$\Rightarrow V_{dip} = k_e\frac{\vec{P}\cdot\hat{r}}{r^2} = \frac{k_e qa}{r^2}(\hat{k}\cdot\hat{r}) = \frac{k_e qa}{r^2}\cos\theta$

$\Rightarrow V(r,\theta) = k_e q\left(\frac{-1}{r} + \frac{a\cos\theta}{r^2}\right) \Rightarrow \vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$

$\Rightarrow \vec{E} = k_e q\left[-\frac{1}{r^2}\hat{r} + \frac{2a\cos\theta}{r^3}\hat{r} + \frac{a}{r^3}\sin\theta\,\hat{\theta}\right]$