Electromagnetic Theory (1)

HW#4 – Solution- Dr. Gassem Alzoubi

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Problem 3.15: A rectangular pipe, running parallel to the z-axis (from −∞ to +∞), has three grounded metal sides, at y = 0, y = a, and x = 0. The fourth side, at x = b, is maintained at a specified potential $V_0(y)$.

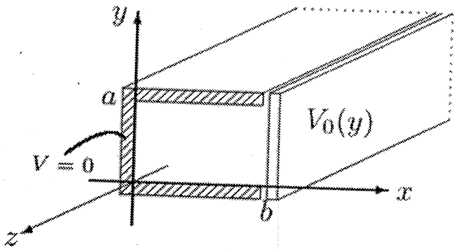
(a) Develop a general formula for the potential inside the pipe.

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

(a) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, with boundary conditions

$V = V(x,y)$ as V is symmetric along the z-axes.

$$\begin{cases} \text{(i)} & V(x,0) = 0, \\ \text{(ii)} & V(x,a) = 0, \\ \text{(iii)} & V(0,y) = 0, \\ \text{(iv)} & V(b,y) = V_0(y). \end{cases}$$



the general solution $V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$

from (i) $V(x,0) = 0 = \underbrace{(Ae^{kx} + Be^{-kx})}_{\text{can't be zero except at } x=\pm\infty \text{, which is outside the interval } 0\leq x\leq b}(C\sin(0) + D)$

$\Rightarrow D = 0$

$\Rightarrow V(x,y) = C(Ae^{kx} + Be^{-kx})\sin ky$, C can be absorbed into A and B

$= (Ae^{kx} + Be^{-kx})\sin ky$

from (iii) $V(0,y) = 0 = (A+B)\underbrace{\sin ky}_{\text{can't be zero}} \Rightarrow A + B = 0 \Rightarrow A = -B$

$\Rightarrow V(x,y) = A(e^{kx} - e^{-kx})\sin ky$; where $\sinh kx = \dfrac{e^{kx} - e^{-kx}}{2}$

$= 2A \sinh kx \sin ky$; let us call it F

$\Rightarrow V(x,y) = F\sinh kx \sin ky$

from (ii) $V(x,a) = 0 = \underbrace{F}\,\underbrace{\sinh kx}_{\text{can't be zero}}\sin ka \Rightarrow \sin ka = 0$

$ka = n\pi$

$k = \dfrac{n\pi}{a}$

$n = 1, 2, 3, 4, \ldots$

$\Rightarrow V(x,y) = F \sinh \frac{n\pi}{a}x \sin \frac{n\pi}{a}y$

the most general solution is

$$V(x,y) = \sum_{n=1}^{\infty} F_n \sinh \frac{n\pi}{a}x \sin \frac{n\pi}{a}y$$, now to find F_n

let us apply B.Cs (iv) $V(b,y) = V_0(y)$

$\Rightarrow \sum_{n=1}^{\infty} F_n \sinh \frac{n\pi}{a}b \sin \frac{n\pi}{a}y = V_0(y)$, multiply by

$\sin \frac{n'\pi}{a}y$ and integrate over dy from $0 \to a$

$$\sum_n F_n \sinh \frac{n\pi}{a}b \underbrace{\int_0^a \sin \frac{n\pi}{a}y \sin \frac{n'\pi}{a}y \, dy}_{a/2 \text{ when } n=n'} = \int_0^a V_0(y) \sin \frac{n'\pi}{a}y \, dy$$

$$F_{n'} \sinh \frac{n'\pi}{a}b \; \frac{a}{2} = \int_0^a V_0(y) \sin \frac{n'\pi}{a}y \, dy \; ; \quad \text{let } n' \to n$$

$$\Rightarrow F_n = \frac{2}{a \sinh \frac{n\pi}{a}b} \int_0^a V_0(y) \sin \frac{n\pi}{a}y \, dy$$

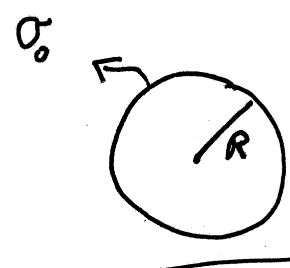
b) when $V_0(y) = V_0$ $\Rightarrow F_n = \frac{2V_0}{a \sinh \frac{n\pi}{a}b} \times \frac{a}{n\pi}[1 - \cos n\pi]$

$$\Rightarrow F_n = \begin{cases} 0, & n \text{ is even} \\ \frac{4V_0}{n\pi \sinh \frac{n\pi}{a}b} ; & n \text{ is odd} \end{cases}$$

$$\Rightarrow V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sinh\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)}{n \sinh\left(\frac{n\pi}{a}b\right)}$$

Problem 3.18: (b) Find the potential inside and outside a spherical shell that carries a uniform surface charge σ_0.

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & r \leq R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & r \geq R \end{cases}$$



B.CS

(i) $V_{in}(r=R) = V_{out}(r=R) \Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow \boxed{B_l = R^{2l+1} A_l}$ —(1)

(ii) $\left.\frac{dV_{out}}{dr}\right|_{r=R} - \left.\frac{dV_{in}}{dr}\right|_{r=R} = -\frac{\sigma_0}{\varepsilon_0}$

$$\Rightarrow \sum_{l=0}^{\infty} -\frac{B_l(l+1)}{R^{l+2}} P_l - l A_l R^{l-1} P_l = -\frac{\sigma_0}{\varepsilon_0}$$

$$\Rightarrow \sum_{l=0}^{\infty} \left[\frac{A_l R^{2l+1}(l+1) P_l}{R^{l+1}} + l A_l R^{l-1} P_l \right\} = +\sigma_0/\varepsilon_0$$

$$\sum_{l=0}^{\infty} \left[A_l R^{l-1}(l+1) P_l + l A_l R^{l-1} P_l \right\} = \sigma_0/\varepsilon_0$$

$$\sum_{l=0}^{\infty} A_l R^{l-1} P_l [l+1+l] = \sigma_0/\varepsilon_0$$

$$\sum_{l=0}^{\infty} A_l R^{l-1}(2l+1) P_l = \sigma_0/\varepsilon_0$$, multiply by $P_{l'} \sin\theta d\theta$ and integrate

$$\sum_{l=0}^{\infty} A_l R^{l-1}(2l+1) \underbrace{\int_0^{\pi} P_{l'} P_l \sin\theta d\theta}_{\frac{2}{2l'+1}\delta_{ll'}} = \frac{\sigma_0}{\varepsilon_0} \int_0^{\pi} P_{l'} \sin\theta d\theta$$

$$A_{l'} R^{l'-1}(2l'+1) \cdot \frac{2}{2l'+1} = \frac{\sigma_0}{\varepsilon_0} \int_0^{\pi} P_{l'} \sin\theta d\theta$$

$$\Rightarrow A_l' = \frac{\sigma_0}{2\varepsilon_0 R^{l'-1}} \int_0^{\pi} P_{l'} \sin\theta \, d\theta \quad , \quad \text{let } l' \to l$$

$$A_l = \frac{\sigma_0}{2\varepsilon_0 R^{l-1}} \int_0^{\pi} P_l(\cos\theta) \sin\theta \, d\theta$$

$$= \frac{\sigma_0}{2\varepsilon_0 R^{l-1}} \int_0^{\pi} \underbrace{P_0(\cos\theta)}_{1} P_l(\cos\theta) \sin\theta \, d\theta$$

$$= \frac{\sigma_0}{2\varepsilon_0 R^{l-1}} \cdot \frac{2}{2l+1} \delta_{l,0} = \begin{cases} \frac{\sigma_0 R}{\varepsilon_0}, & l = 0 \\ 0, & l \neq 0 \end{cases}$$

so only A_0 servive ; $A_0 = \frac{\sigma_0 R}{\varepsilon_0}$

$$V(r,\theta) = \begin{cases} A_0 P_0 = \frac{\sigma_0 R}{\varepsilon_0}, & r \leq R \\ \frac{B_0}{r} P_0 = \underbrace{\frac{B_0}{r} = \frac{R A_0}{r}}_{\text{from (1)}} = \frac{\sigma_0 R^2}{\varepsilon_0} \frac{1}{r}, & r \geq R \end{cases}$$

Now using $Q = 4\pi R^2 \sigma_0$

$$\Rightarrow V(r,\theta) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{Q}{R}, & r \leq R \\ \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}, & r \geq R \end{cases}$$

Problem 3.19

The potential at the surface of a sphere is given by $V_o(\theta) = k\cos(3\theta)$ where k is some constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume that there is no charge inside or outside of the sphere.)

the most general solution is $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

i) $r < R$ (inside);

here $B_l = 0$ since otherwise $V(r,\theta)$ would blow up at $r=0$

$\Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

at $r=R$, $V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = k\cos 3\theta$

let us express $k\cos\theta$ in terms of Legendre polynomials; and $P_1(x) = x$

$P_3(x) = \frac{5x^3 - 3x}{2}$; when $x = \cos\theta$

$\Downarrow$

$5x^3 - 3x = 2P_3 \Rightarrow x^3 = \frac{2}{5}P_3 + \frac{3}{5}x = \frac{2}{5}P_3 + \frac{3}{5}x$

$\Rightarrow 4x^3 = \frac{8}{5}P_3 + \frac{12}{5}x \underset{\text{but}}{\Rightarrow} \boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$ identity

so $\cos 3\theta = 4x^3 - 3x = \frac{8}{5}P_3 + \frac{12}{5}x - 3x = \frac{8}{5}P_3 + \left(\frac{12}{5} - 3\right)x$

$= \frac{8}{5}P_3 - \frac{3}{5}x = \frac{8}{5}P_3 - \frac{3}{5}P_1$ as $P_1 = x$

$\therefore \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \frac{8}{5}kP_3 - \frac{3}{5}kP_1$;

equating coefficients, only $l=1$ and $l=3$ survive

$A_0 P_0(\cos\theta) + A_1 R P_1(\cos\theta) + A_2 R^2 P_2(\cos\theta) + A_3 R^3 P_3(\cos\theta)$

$+ A_4 R^4 P_4(\cos\theta) + \cdots = \frac{8}{5}kP_3(\cos\theta) - \frac{3}{5}kP_1(\cos\theta)$

so $A_1 R = -\frac{3}{5}k \Rightarrow A_1 = \frac{-3k}{5R}$ and $A_3 R^3 = \frac{8}{5}k$

$\Rightarrow A_3 = \frac{8k}{5R^3}$

So the potential inside the sphere is

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta)$$

$$= -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta)$$

$$\boxed{V(r,\theta) = -\frac{3k}{5R} r\cos\theta + \frac{8k}{5R^3} r^3 \cdot \frac{1}{2}(5\cos^3\theta - 3\cos\theta)} \quad r<R$$

ii) $r>R$ (outside); here $A_l = 0$, since otherwise V would blow up at ∞, so

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

at $r=R$, we have

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = \frac{8k}{5}P_3 - \frac{3k}{5}P_1$$

again, only two terms servive ($l=1$ and $l=3$)

$$\Rightarrow B_1 = -\frac{3k}{5}R^2 \text{ and } B_3 = \frac{8k}{5}R^4, \text{ so}$$

$$V(r,\theta) = \frac{B_1}{r^2}P_1(\cos\theta) + \frac{B_3}{r^4}P_3(\cos\theta)$$

$$= -\frac{3k}{5}\frac{R^2}{r^2}P_1(\cos\theta) + \frac{8k}{5}\frac{R^4}{r^4}P_3(\cos\theta)$$

$$\boxed{V(r,\theta) = -\frac{3k}{5}R^2\frac{1}{r^2}\cos\theta + \frac{8k}{5}R^4\frac{1}{r^4}\cdot\frac{1}{2}(5\cos^3\theta - 3\cos\theta)} \quad r>R$$

$\sigma(\theta)$ can be found by applying B.Cs of $\vec{E}$ at $r=R$,

$$\vec{E}_{r=R^+} - \vec{E}_{r=R^-} = \frac{\sigma(\theta)}{\varepsilon_0}\hat{r} \quad ; \text{ but } \vec{E} = -\vec{\nabla}V$$

$$(-\nabla V)^{out}_{r=R^+} - (-\nabla V)^{in}_{r=R^-} = \frac{\sigma}{\varepsilon_0}$$

$$(\nabla V)^{out}_{r=R} - (\nabla V)^{in}_{r=R} = -\frac{\sigma}{\varepsilon_0}$$

$$\left(\frac{\partial V}{\partial r}\right)^{out}_{r=R} - \left(\frac{\partial V}{\partial r}\right)^{in}_{r=R} = -\frac{\sigma}{\varepsilon_0} \quad --- (*)$$

now

$$\left(\frac{\partial V}{\partial r}\right)^{out}_{r=R} = \left[\frac{6k}{5}\frac{R^2}{r^3}P_1 - \frac{32k}{5}\frac{R^4}{r^5}P_3\right]_{r=R}$$

$$= \frac{k}{5R}(6P_1 - 32P_3)$$

similarly

$$\left(\frac{\partial V}{\partial r}\right)^{in}_{r=R} = -\frac{3k}{5}\frac{1}{R}P_1 + \frac{24k}{5}\frac{r^2}{R^3}P_3$$

$$= \frac{k}{5R}(-3P_1 + 24P_3) \quad ; \text{ substitute in } (*)$$

$$\Rightarrow \left(\frac{\partial V}{\partial r}\right)^{out}_{r=R} - \left(\frac{\partial V}{\partial r}\right)^{in}_{r=R} = -\frac{\sigma}{\varepsilon_0}$$

$$\frac{k}{5R}(6P_1 - 32P_3) - \frac{k}{5R}(-3P_1 + 24P_3) = -\frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \sigma(\theta) = \frac{k\varepsilon_0}{5R}\left(-9P_1(\cos\theta) + 56P_3(\cos\theta)\right)$$

$$= \frac{k\varepsilon_0}{5R}\left(-9\cos\theta + 56\cdot\frac{1}{2}(5\cos^3\theta - 3\cos\theta)\right)$$

Problem 3.24: Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find all solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

in cylindrical coordinate $x = s\cos\phi,\ y = s\sin\phi,$ and $z = z$

$P(s, \phi, z)$

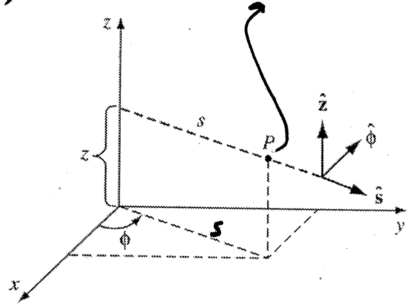


FIGURE 1.42

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{k}$$

$$\Rightarrow V = V(s, \phi, z) \Rightarrow \nabla^2 V = 0$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

No z dependence $\Rightarrow \dfrac{\partial^2 V}{\partial z^2} = 0 \Rightarrow$ and $V = V(s, \phi)$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0\ ,\quad \text{let } V = S(s)\,\Phi(\phi)$$

substitute V in the above eq^n, multiply by s^2, and divide by $V = S\Phi$

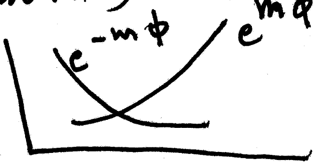
$$\Rightarrow \underbrace{\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)}_{C_1} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{-C_1} = 0$$

First consider negative value of C_1 (set $C_1 = -m^2 < 0$)

$$\Rightarrow \frac{d^2\Phi}{d\phi^2} - m^2\Phi = 0 \Rightarrow \Phi_m(\phi) = A_m \sinh m\phi + B_m \cosh m\phi$$

or $= C_m e^{m\phi} + D_m e^{-m\phi}$

but this is not a satisfactory solution because it is not a periodic solution as it is required that in cylindrical symmetry, that Φ must be periodic (single-valued) in ϕ. Note the behavior of $e^{\pm m\phi}$ terms



, so we conclude that C_1 can't be negative

Note also that if ϕ increases by steps of 2π, $\Phi_m(\phi)$ is either increasing ($e^{m\phi}$) or decreasing ($e^{-m\phi}$), resulting in $\Phi_m(\phi)$ being Not Periodic (single-valued.).

Second, consider $C_1 = m^2 > 0 \Rightarrow \frac{d^2\Phi}{d\phi^2} + m^2 \Phi = 0$

$\Rightarrow \Phi_m(\phi) = C_m \cos(m\phi) + D_m \sin(m\phi)$

or $\approx e^{\pm i m \phi}$ $\longrightarrow$ Periodic solution (oscillatory)

Note that Φ_m is Periodic as $\Phi_m(\phi+2\pi) = \Phi_m(\phi)$

$\Rightarrow e^{im(\phi+2\pi)} = e^{im\phi} \Rightarrow \cancel{e^{im\phi}} e^{i2\pi m} = \cancel{e^{im\phi}}$

$\Rightarrow e^{i2\pi m} = 1 \Rightarrow \cos(2\pi m) + i\sin(2\pi m) = 1 \Rightarrow \cos(2\pi m) = 1$

$\Rightarrow m = 0, 1, 2, 3, \ldots$ only positive values. Note that considering $e^{-im(\phi+2\pi)} = e^{-im\phi} \Rightarrow$ will gives $e^{-i2\pi m} = 1$

$\Rightarrow \cos(2\pi m) - i\sin 2\pi m = 1 \Rightarrow \cos(2\pi m) = 1 \Rightarrow m = 0, 1, 2, 3, \ldots$

Now for the radial eq^n

$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = m^2 S$, try solution $S(s) = As^k$

$\Rightarrow s\frac{d}{ds}\left(s\,Aks^{k-1}\right) = m^2 \cancel{A}s^k \Rightarrow ks\frac{d}{ds}(s^k) = m^2 s^k$

$\Rightarrow ksks^{k-1} = m^2 s^k \Rightarrow k^2 \cancel{s^k} = m^2 \cancel{s^k} \Rightarrow k^2 = m^2$

$\Rightarrow k = \pm m$

$$\Rightarrow S(s) = A_m s^m + B_m s^{-m}$$

$$= A_m s^m + \frac{B_m}{s^m}$$

Now consider the case of $m^2 = 0$, then the radial equation reads

$$\frac{s}{\mathcal{S}}\frac{d}{ds}\left(s\frac{d\mathcal{S}}{ds}\right) = c_1 = m^2 = 0 \Rightarrow \frac{d}{ds}\left(s\frac{d\mathcal{S}}{ds}\right) = 0$$

$$\Rightarrow s\frac{d\mathcal{S}}{ds} = a_0 = \text{constant} \Rightarrow \frac{d\mathcal{S}}{ds} = \frac{a_0}{s}$$; integrate

$$\mathcal{S}(s) = a_0 \ln s + b_0$$

Now combining the solutions obtained for $m^2 = 0$ with solutions obtained for $m^2 > 0$, we conclude that the most general solution for the radial eq^n is

$$\mathcal{S}(s) = a_0 \ln s + b_0 + \sum_{m=1}^{\infty}\left[A_m s^m + \frac{B_m}{s^m}\right]$$

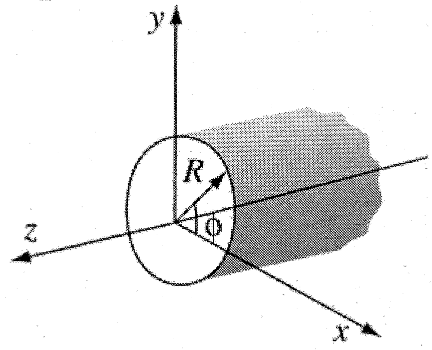
therefore, the most general solution for Laplace's equation can be written as

$$V(s,\phi) = a_0 \ln(s) + b_0 + \sum_{m=1}^{\infty}\left[\left(A_m s^m + \frac{B_m}{s^m}\right)\left(C_m \cos(m\phi) + D_m \sin(m\phi)\right)\right]$$

Problem 3.26

Charge density $\sigma(\phi) = a\sin(5\phi)$ (where a is a constant) is glued over the surface of an infinite cylinder of radius R. Find the potential inside and outside the cylinder. [Use your result from Prob. 3.24.]

from prob 3.24, we can write the Potential as



inside $(s<R)$; in this region $a_0 \ln s$ and $\frac{B_m}{s^m}$ are Not a good at $s=0$; blow up at $s=0$

$\Rightarrow$ set $a_0 = 0$, $B_m = 0$

$$\Rightarrow V_{in}(s,\phi) = b_0 + \sum_{m=1}^{\infty} A_m s^m (C_m \cos(m\phi) + D_m \sin(m\phi))$$

A_m can be absorbed in C_m and D_m and expressed into new constants a_m and b_m, so

$$V_{in}(s,\phi) = b_0 + \sum_{m=1}^{\infty} s^m (a_m \cos(m\phi) + b_m \sin(m\phi))$$

outside $(s>R)$: in this region $A_m s^m$ and $a_0 \ln s$ are Not good terms as they blow up at ∞

so set $a_0 = 0$ and $A_m = 0$

$$V_{out}(s,\phi) = b_0' + \sum_{m=1}^{\infty} \frac{B_m'}{s^m} (C_m' \cos(m\phi) + D_m' \sin(m\phi))$$

again B_m' can be absorbed into C_m' and D_m' and relabeled them to C_R and d_R, so

$$V_{out}(s,\phi) = b_0' + \sum_{m=1}^{\infty} \frac{1}{s^m} (c_m \cos(m\phi) + d_m \sin(m\phi))$$

Now the discontinuity in $\vec{E}$ gives

$$\vec{E}_{out} - \vec{E}_{in} = \frac{\sigma}{\varepsilon_0} \Rightarrow \sigma = \varepsilon_0 \left(\vec{E}_{out} - \vec{E}_{in} \right)$$

$$\sigma = \varepsilon_0 \left(-\nabla V_{out} - (-\nabla V_{in}) \right) = -\varepsilon_0 \left(\frac{\partial V_{out}}{\partial s}\bigg|_{s=R} - \frac{\partial V_{in}}{\partial s}\bigg|_{s=R} \right)$$

thus

$$a \sin(5\phi) = -\varepsilon_0 \sum_{m=1}^{\infty} \left[-\frac{m}{R^{m+1}} \left(c_m \cos(m\phi) + d_m \sin(m\phi) \right) - m R^{m-1} \left(a_m \cos(m\phi) + b_m \sin(m\phi) \right) \right]$$

equating coefficients on both sides, indicates that only one term servive on RHS (when $m = 5$)

where $c_m = a_m = 0$ for all m

$b_m = d_m = 0$ except for $m = 5$, so

$$a \sin(5\phi) = \frac{5\varepsilon_0}{R^6} \left(\cancelto{0}{c_5} \cos(5\phi) + d_5 \sin(5\phi) \right) + 5\varepsilon_0 R^4 \left(\cancelto{0}{a_5} \cos(5\phi) + b_5 \sin(5\phi) \right)$$

$$a \sin(5\phi) = \frac{5\varepsilon_0}{R^6} d_5 \sin(5\phi) + 5\varepsilon_0 R^4 b_5 \sin(5\phi)$$

$$\Rightarrow a = \frac{5\varepsilon_0}{R^6} d_5 + 5\varepsilon_0 R^4 b_5 \Rightarrow \boxed{a = 5\varepsilon_0 \left(\frac{d_5}{R^6} + R^4 b_5 \right)} \quad --- (*)$$

the potentials inside and outside reads

$$V_{in}(s, \phi) = b_0 + s^5 (b_5 \sin(5\phi)) \qquad s < R$$

$$V_{out}(s, \phi) = b_0' + \frac{1}{s^5} d_5 \sin(5\phi) \qquad s > R$$

but V is contineous across the surface ($s = R$), so

$$b_0 + R^5 b_5 \sin(5\phi) = b_0' + \frac{1}{R^5} d_5 \sin(5\phi)$$

we see that $b_0 = b_0'$ and can be set to zero

and $R^5 b_5 = \frac{1}{R^5} d_5 \Rightarrow \boxed{d_5 = R^{10} b_5} \quad --- (**)$

solving (*) and (**) for b_5 and d_5, we get

$$b_5 = \frac{a}{10\varepsilon_0 R^4} \quad \text{and} \quad d_5 = \frac{aR^6}{10\varepsilon_0}$$

so the potentials now reads (and setting $b_0 = b_0' = 0$)

$$V_{in}(s,\phi) = b_5 s^5 \sin(5\phi) = \frac{a}{10\varepsilon_0 R^4} s^5 \sin(5\phi)$$

and

$$V_{out} = \frac{d_5}{s^5} \sin(5\phi) = \frac{aR^6}{10\varepsilon_0} \frac{1}{s^5} \sin(5\phi)$$

$$\therefore V(s,\phi) = \frac{a\sin(5\phi)}{10\varepsilon_0} \begin{cases} \dfrac{s^5}{R^4} & ; \text{for } s < R \\ \dfrac{R^6}{s^5} & ; \text{for } s > R \end{cases}$$