Electromagnetic theory (1)

HW#3-Solution – Dr. Gassem Alzoubi

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Problem 3.7: Find the force on the charge +q in Fig. 3.14. (The xy plane is a grounded conductor.)

the force on the real charge $+q$ is the net force due to the real charge $(-2q)$ and the two image charges $(+2q)$ and $(-q)$, so

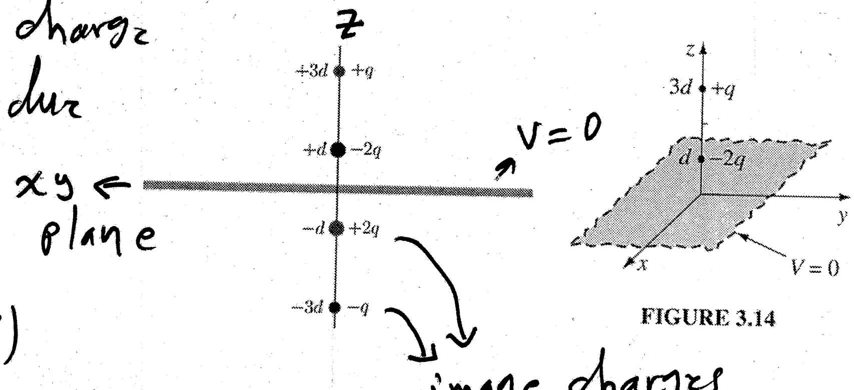
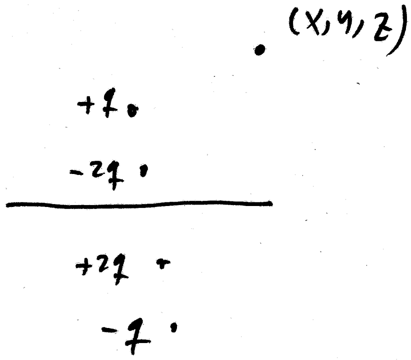


FIGURE 3.14

$$\vec{F}_q = k_e \frac{q(-2q)}{(2d)^2}\hat{k} + k_e \frac{q(+2q)}{(4d)^2}\hat{k} + k_e \frac{q(-q)}{(6d)^2}\hat{k}$$

$$= \frac{k_e q^2}{d^2}\left[-\frac{2}{4} + \frac{2}{16} - \frac{1}{36}\right]\hat{k} = \frac{k_e q^2}{d^2}\left[-\frac{29}{72}\right]\hat{k}$$

$$= -\frac{29}{72}\frac{k_e q^2}{d^2}\hat{k} \text{ , downward}$$



Note that $V(x,y,z) = V_{+q} + V_{-2q} + V_{+2q} + V_{-q}$

$$V(x,y,z) = k_e\left[\frac{q}{\sqrt{x^2+y^2+(z-3d)^2}} - \frac{2q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{2q}{\sqrt{x^2+y^2+(z+2d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+3d)^2}}\right]$$

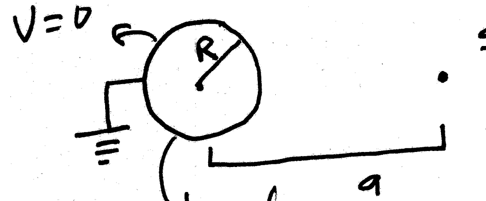
$\sigma = -\varepsilon_0 \dfrac{dV}{dz} \Rightarrow q_{induced} = \int \sigma\, da \equiv$ must be the sum of image charges $= +2q - q = +q$

on the plane

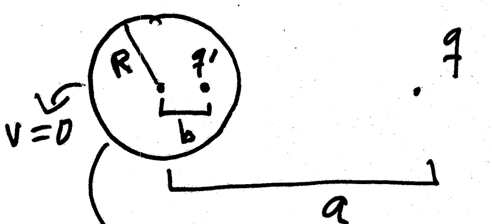
Problem 3.9: In example 3.2 we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at any potential V_0 (relative, of course, to infinity). What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a neutral conducting sphere.

the original problem was a point charge $(+q)$ next to a grounded metal sphere.

The equivalent image problem was a real charge $(+q)$ plus an imaginary charge q' placed at $r = b$ inside an imaginary spherical surface (Not a conductor)



Real conducting sphere

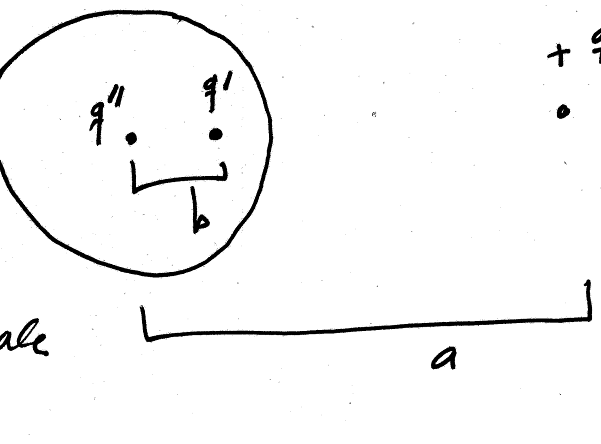


with $b = R^2/a$; $q' = -\frac{R}{a} q$

this is not a conductor, it is just an imaginary spherical surface with radius (R)

Now we want to increase the potential on the metallic sphere to V_0. Note that the sphere is not grounded now. This can be done by placing a positive charge on the sphere or equivalently by placing a second image charge (q'') placed at the center of the imaginary spherical surface as shown in figure.

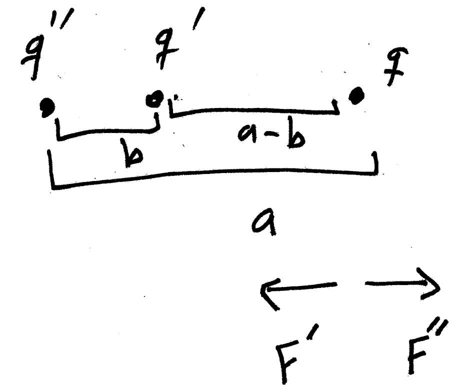
This is not a spherical conductor. it is just a surface



so $V_0 = k_e \dfrac{q''}{R} \Rightarrow q'' = \dfrac{V_0 R}{k_e} = 4\pi \varepsilon_0 V_0 R$

Now for neutral sphere, $q' + q'' = 0 \Rightarrow q'' = -q' = \dfrac{R}{a} q$,

so the force exerted by the neutral sphere on the outside charge (+q) is the same as the force exerted by the two image charges (q', q'') on the charge (+q), so



$$F_q = k_e \frac{q'' q}{a^2} + k_e \frac{q' q}{(a-b)^2}$$

$$= k_e \frac{R}{a} \frac{q^2}{a^2} - k_e \frac{R}{a} \frac{q^2}{(a-b)^2}$$

$$= k_e q^2 \left(\frac{R}{a}\right)\left[\frac{1}{a^2} - \frac{1}{(a-b)^2}\right] = k_e q^2 \left(\frac{R}{a}\right)\left[\frac{(a-b)^2 - a^2}{a^2(a-b)^2}\right]$$

$$= k_e q^2 \left(\frac{R}{a}\right)\left[\frac{a^2 + b^2 - 2ab - a^2}{a^2(a-b)^2}\right] = k_e q^2 \left(\frac{R}{a}\right) \frac{b^2 - 2ab}{a^2 (a-b)^2}$$

$$= k_e q^2 \left(\frac{R}{a}\right)\left[\frac{\frac{R^4}{a^2} - 2R^2}{a^2 \left(a - \frac{R^2}{a}\right)^2}\right] = k_e q^2 \left(\frac{R}{a}\right)^3 \left[\frac{\frac{R^2}{a^2} - 2}{\left(a - \frac{R^2}{a}\right)^2}\right]$$

$$= k_e q^2 \left(\frac{R}{a}\right)^3 \left[\frac{\frac{R^2 - 2a^2}{\cancel{a^2}}}{\frac{1}{\cancel{a^2}}(a^2 - R^2)^2}\right] = k_e q \left(\frac{R}{a}\right)^3 \left[\frac{R^2 - 2a^2}{(a^2 - R^2)^2}\right]$$

and points to left as $R < a$ always

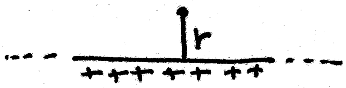
Problem 3.10 A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

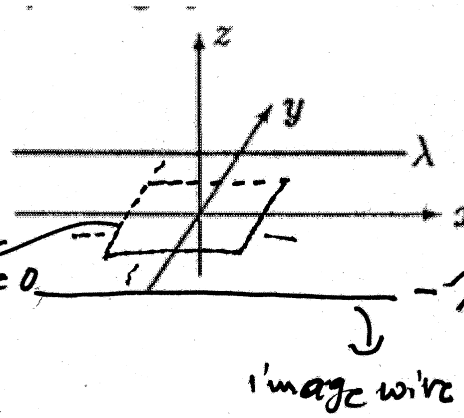
(a) Find the potential in the region above the plane. refer to prob 2.52

(b) Find the charge density σ induced on the conducting plane

for an infinite wire with λ, $\vec{E}$ is

$$E_r = \frac{2k_e\lambda}{r} \; ;$$





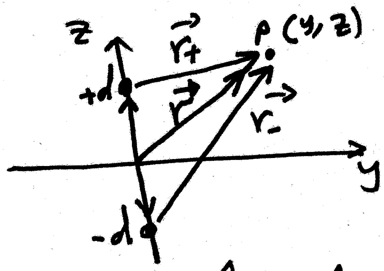
$$V(r) = -\int \vec{E}\cdot d\vec{l} = -\int E_r dr = -\int \frac{2k_e\lambda}{r} dr$$

$$= -2k_e\lambda \int \frac{dr}{r} = -2k_e\lambda \ln r$$

$$V_p = V_+ + V_- = -2k_e\lambda \ln r_+ - 2k_e(-\lambda)\ln r_-$$

$$= -2k_e\lambda \ln r_+ + 2k_e\lambda \ln r_- = 2k_e\lambda \ln\left(\frac{r_-}{r_+}\right)$$

$$= k_e\lambda \ln\left(\frac{r_-}{r_+}\right)^2 = k_e\lambda \ln\left(\frac{r_-^2}{r_+^2}\right)$$

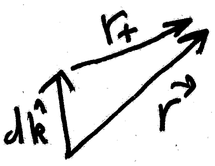


but $\vec{r} = d\hat{k} + \vec{r}_+ \Rightarrow \vec{r}_+ = \vec{r} - d\hat{k}$

$$= y\hat{j} + z\hat{k} - d\hat{k}$$

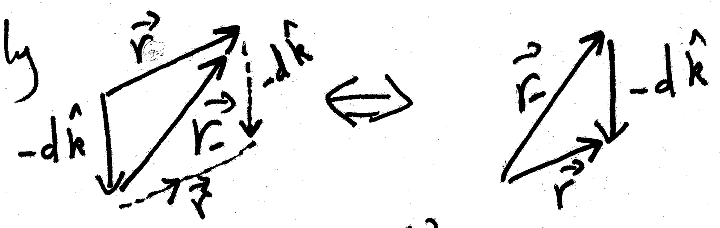
$$= y\hat{j} + (z-d)\hat{k}$$

$$\vec{r} = y\hat{j} + z\hat{k}$$



$$\Rightarrow r_+^2 = y^2 + (z-d)^2$$

similarly



$\Rightarrow \vec{r} = \vec{r}_- - d\hat{k}$

$\Rightarrow \vec{r}_- = \vec{r} + d\hat{k}$

$$= y\hat{j} + z\hat{k} + d\hat{k}$$

$$= y\hat{j} + (z+d)\hat{k}$$

$$\Rightarrow r_-^2 = y^2 + (z+d)^2$$

Notice that the potential is invariant under the change of x, so we can set $x=0$

$$\Rightarrow V_p = k_e\lambda \ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right] = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]$$

b) at a point very close to the plane,

$$E_z = -\frac{dV}{dz} = \frac{\sigma}{\varepsilon_0} \Rightarrow \sigma = -\varepsilon_0 \frac{dV}{dz}\Big|_{z=0} \quad ; \text{when } \sigma = \sigma(y) \text{ only}$$

$$V_p = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(y^2+(z+d)^2\right) - \ln\left[y^2+(z-d)^2\right]\right]$$

$$\sigma(y) = -\varepsilon_0 \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right]_{z=0}$$

$$= -\frac{\lambda}{2\pi}\left[\frac{z+d}{y^2+(z+d)^2} - \frac{z-d}{y^2+(z-d)^2}\right]_{z=0}$$

$$= -\frac{\lambda}{2\pi}\left\{\frac{d}{y^2+d^2} + \frac{d}{y^2+d^2}\right\} = -\frac{\lambda}{2\pi}\,\frac{2d}{y^2+d^2}$$

$$= -\frac{\lambda d}{\pi(y^2+d^2)}$$

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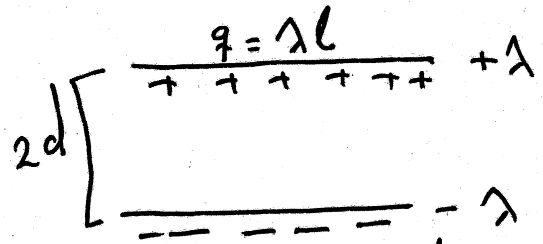
c) extra: what is the force exerted by the induced charge on the plane on the line charge λ (per unit length $\frac{F}{l}$).
This force is the same as the mutual force between the two line charge distribution ($+\lambda$ and $-\lambda$),

$$\vec{F}_{-\lambda \text{ on } +\lambda} = q\vec{E}_{-\lambda} \quad ; \text{where } E_{-\lambda} = -\frac{2k_e\lambda}{2d}\hat{k}$$

$$= (\lambda l)\left(-\frac{2k_e\lambda}{2d}\right)\hat{k}$$

$$\Rightarrow \quad \frac{\vec{F}}{l} = -\frac{k_e\lambda^2}{d}\hat{k}$$



$E_{-\lambda}$ is the field created by $-\lambda$ at the position of $+\lambda$ line