HW#2- Solutions- Dr. Gassem Alzoubi

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Problem 2.20 One of these is an impossible electrostatic field. Which one?

(a) $\mathbf{E}_1 = k[xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3xz\,\hat{\mathbf{z}}]$;

(b) $\mathbf{E}_2 = k[y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}]$.

Here k is a constant with the appropriate units. For the *possible* one, find the potential, using the *origin* as your reference point. Check your answer by computing ∇V. [*Hint:* You must select a specific path to integrate along. It doesn't matter *what* path you choose, since the answer is path-independent, but you simply cannot integrate unless you have a definite path in mind.]

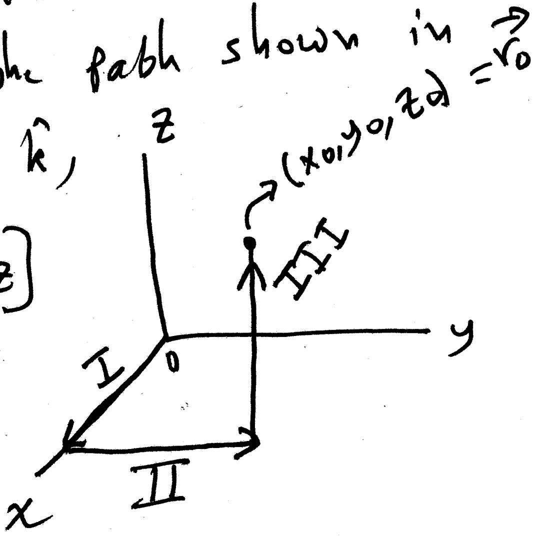
a) $\nabla \times \vec{E}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = k[-2y\hat{i} - 3z\hat{j} - x\hat{k}] \neq 0$

so E_1 is impossible to be electrostatic field.

b) $\nabla \times \vec{E}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix} = k[0] = 0$, so

E_2 is a possible electrostatic field.

so to find V_2, let us pick up the path shown in figure. Now $d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$,



so $\vec{E}_2 \cdot d\vec{l} = k[y^2 dx + (2xy + z^2)dy + 2yz\,dz]$

$\Rightarrow V(\vec{r}_0) - V(0) = -\int_0^{r_0} \vec{E}_2 \cdot d\vec{l}$

Reference point, where $V(0) = 0$ as $E_2(0) = 0$

$\Rightarrow V(x_0, y_0, z_0) = -\int_0^{x_0, y_0, z_0} \vec{E}_2 \cdot d\vec{l}$

$$\Rightarrow V(x_0,y_0,z_0) = -\int_{\mathrm{I}} \vec{E_2}\cdot d\vec{l} - \int_{\mathrm{II}} \vec{E_2}\cdot d\vec{l} - \int_{\mathrm{III}} \vec{E_2}\cdot d\vec{l}$$

Path I: $y = z = 0,\ dy = dz = 0 \Rightarrow \int_{\mathrm{I}} \vec{E_2}\cdot d\vec{l} = k\int_{0,\,y=0}^{x_0} y^2\,dx = 0$

path II: $x = x_0,\ z = 0,\ dx = dz = 0 \Rightarrow$

$$\int_{\mathrm{II}} \vec{E_2}\cdot d\vec{l} = k\int (2xy + z^2)\,dy = 2k x_0 \int_0^{y_0} y\,dy = k x_0 y_0^2$$

Path III : $x = x_0,\ y = y_0,\ dx = dy = 0 \Rightarrow$

$$\int_{\mathrm{III}} \vec{E_2}\cdot d\vec{l} = k\int_{0,\,y=y_0}^{z_0} 2yz\,dz = 2ky_0 \int z\,dz = k y_0 z_0^2$$

$\Rightarrow V(x_0,y_0,z_0) = -k(x_0 y_0^2 + y_0 z_0^2)$, so in general at any point, we can write $V(x,y,z) = -k(xy^2 + yz^2)$

let us check that $\vec{E_2} = -\nabla V$

$$-\nabla V = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(-kxy^2 - kyz^2)$$

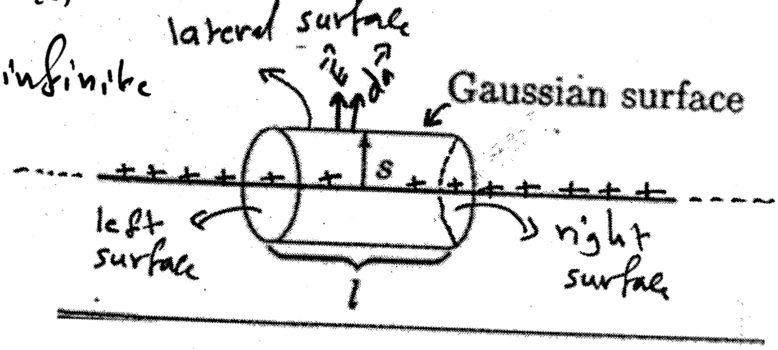
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)k(xy^2 + yz^2)$$

$$= k\left[y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}\right) \equiv \vec{E_2}$$

as expected

Problem 2.22: Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

from problem 2.13, we found $\vec{E}$ a distance s from an infinite line that carries a uniform charge λ



$$\vec{E} = \frac{2k_e\lambda}{s}\hat{s}$$, so

in this case we can not set the reference point at $s=0$ or $s=\infty$ since there is a charge at both ends. let us take V to be zero at an arbitrary point $s=a$, so

$$\underbrace{V(s) - V(a)}_{\text{zero}} = -\int \vec{E}\cdot d\vec{l} \quad ; \text{where } d\vec{l} = ds\,\hat{s}$$

$$= -\int_a^s \frac{2k_e\lambda}{s}\,ds = -2k_e\lambda\int_a^s \frac{ds}{s}$$

$$= -2k_e\lambda \ln s\Big|_a^s = -2k_e\lambda\left[\ln s - \ln a\right]$$

$$= -2k_e\lambda \ln\left(\frac{s}{a}\right)$$

$\Rightarrow V(s) = -2k_e\lambda\ln\left(\frac{s}{a}\right)$; we see here that why $s=0, a$ are not a good ref points

to check ??

$$-\nabla V = -\frac{\partial V}{\partial s}\hat{s}$$

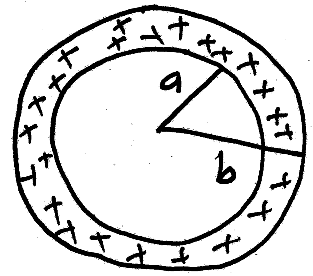
$$= 2k_e\lambda\frac{\partial}{\partial s}\left(\ln\left(\frac{s}{a}\right)\right)\hat{s} = 2k_e\lambda\,\frac{\frac{1}{\cancel{a}}}{\frac{s}{\cancel{a}}}\hat{s} = \frac{2k_e\lambda}{s}\hat{s}$$

as expected.

Problem 2.23: For the charge configuration of Prob. 2.15, find the potential at the center, using infinity as your reference point

This a thick spherical shell with charge density

$$\rho(r) = \frac{k}{r^2} \quad (a \leq r \leq b)$$



from problem 2.15,

$$\vec{E} = \begin{cases} 0, & r<a \\ \frac{k}{\varepsilon_0}\left(\frac{r-a}{r^2}\right)\hat{r}, & a<r<b \\ \frac{k}{\varepsilon_0}\left(\frac{b-a}{r^2}\right)\hat{r}, & r>b \end{cases} \quad \text{, so}$$

$$V(0) - V(\infty) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{l} \quad ; \quad d\vec{l} = dr\,\hat{r} \text{ , in spherical coordinates}$$

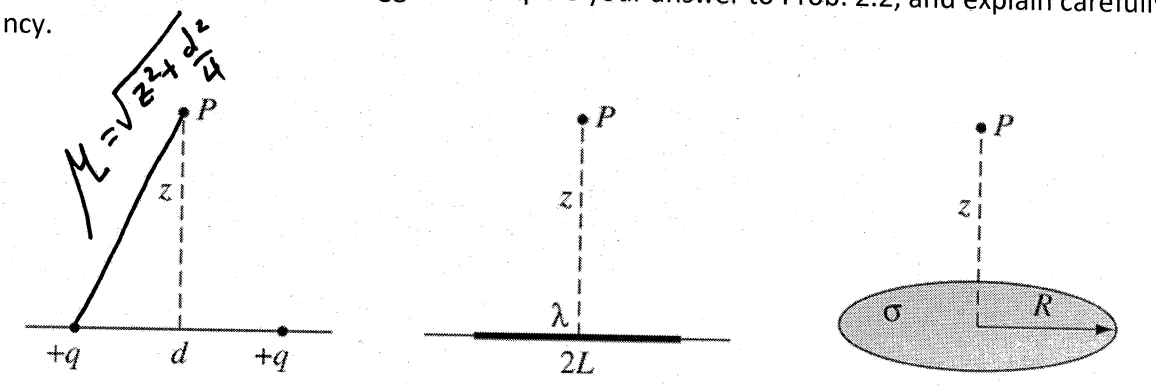
$$= -\int_{\infty}^{b} \frac{k}{\varepsilon_0}\left(\frac{b-a}{r^2}\right) dr - \int_{b}^{a} \frac{k}{\varepsilon_0}\left(\frac{r-a}{r^2}\right) dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{k}{\varepsilon_0}\frac{(b-a)}{b} - \frac{k}{\varepsilon_0}\left[\ln\left(\frac{a}{b}\right) + a\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

$$= \frac{k}{\varepsilon_0}\left\{\cancel{1} - \cancel{\frac{a}{b}} - \ln\left(\frac{a}{b}\right) - \cancel{1} + \cancel{\frac{a}{b}}\right\}$$

$$= -\frac{k}{\varepsilon_0}\ln\left(\frac{a}{b}\right) = \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

Problem 2.25: Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $E = -\nabla V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$; what then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.

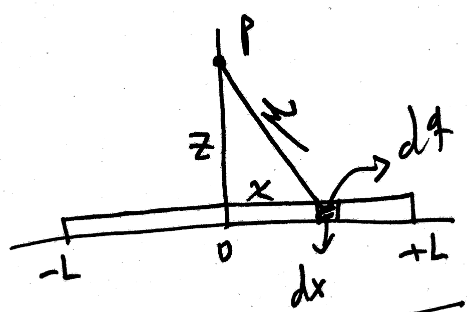


(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

FIGURE 2.34

a) $v = 2k_e \frac{q}{M} = 2k_e \frac{q}{\sqrt{z^2+\frac{d^2}{4}}} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{\sqrt{z^2+\frac{d^2}{4}}}$

b) $dv = k_e \frac{dq}{M} = k_e \frac{\lambda\, dx}{\sqrt{z^2+x^2}}$



$v = k_e \lambda \int_{-L}^{L} \frac{dx}{\sqrt{z^2+x^2}}$;

$M = \sqrt{x^2+z^2}$

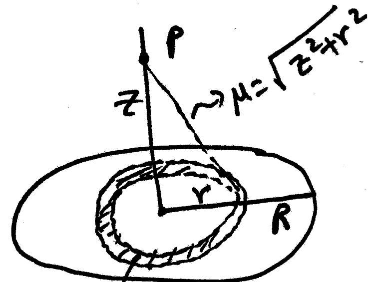
use integral calculator

$v = k_e\lambda \left[\ln(x+\sqrt{z^2+x^2}) - \ln z\right]_{-L}^{+L}$

$= \frac{\lambda}{4\pi\varepsilon_0}\left[\ln(L+\sqrt{z^2+L^2}) - \cancel{\ln z} - \ln(-L+\sqrt{z^2+L^2}) + \cancel{\ln z}\right]$

$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right]$

c) $dV = k_e \dfrac{dq}{\mu} = k_e \dfrac{\sigma\, dA}{\sqrt{z^2+r^2}}$



$$= k_e \frac{\sigma\, 2\pi r\, dr}{\sqrt{z^2+r^2}}$$

$dq = \sigma\, dA$
$= \sigma\, 2\pi r\, dr$

$\Rightarrow V = k_e \sigma \pi \displaystyle\int_0^R \frac{2r\, dr}{\sqrt{z^2+r^2}}$; let $u = z^2+r^2$
$du = 2r\,dr$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0} \sqrt{r^2+z^2}\,\Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - z\right]$$

Now to find fields, use $\vec{E} = -\nabla V$

a) $\vec{E} = -\nabla V = -\left(\dfrac{\partial}{\partial x}\hat{i} + \dfrac{\partial}{\partial y}\hat{j} + \dfrac{\partial}{\partial z}\hat{k}\right) V$

$$= -\left[\underbrace{\frac{\partial V}{\partial x}\hat{i}}_{\text{zero}} + \underbrace{\frac{\partial V}{\partial y}\hat{j}}_{\text{zero}} + \frac{\partial V}{\partial z}\hat{k}\right] = -\frac{dV}{dz}\hat{k}$$

$$= -\frac{2q}{4\pi\varepsilon_0}\frac{d}{dz}\left(z^2+\frac{d^2}{4}\right)^{-1/2}\hat{k}$$

$$= -\frac{2q}{4\pi\varepsilon_0}\left(-\tfrac{1}{\cancel{2}}\right)\left(z^2+\frac{d^2}{4}\right)^{-3/2}(\cancel{2}z)\,\hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left(z^2+d^2/4\right)^{3/2}}\hat{k} \quad \text{as expected.}$$

b) similarly $\vec{E} = -\nabla V = -\dfrac{dV}{dz}\hat{k} = \dfrac{2L\lambda}{4\pi\varepsilon_0}\dfrac{1}{z\sqrt{z^2+L^2}}\hat{k}$

c) similarly $\vec{E} = -\nabla V = -\dfrac{dV}{dz}\hat{k} = \dfrac{\sigma}{2\varepsilon_0}\left(1 - \dfrac{z}{\sqrt{R^2+z^2}}\right)\hat{k}$

Problem 2.34: Find the energy stored in a uniformally charged solid sphere of radius R and charge q?

method 1: $W = \frac{1}{2}\int \rho V \, d\tau$; integration is over the region where charge is located $0 \to R$

from problem 2.21, we found

$$V(r) = \frac{k_e q}{2R}\left(3 - \frac{r^2}{R^2}\right) ; \quad r < R$$

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$\Rightarrow W = \frac{k_e q \rho}{4R}\int \left(3 - \frac{r^2}{R^2}\right) d\tau ; \quad \text{and } \rho = \frac{q}{V} = \frac{q}{\frac{4}{3}\pi R^3}$$

$$= \frac{k_e q \rho}{4R}\int_0^R dr \, r^2\left(3 - \frac{r^2}{R^2}\right) \underbrace{\int_0^\pi \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= k_e \frac{3}{5} \frac{q^2}{R}$$

method 2: $W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\\text{space}}} E^2 d\tau$; integration is over all space

$$= \frac{\varepsilon_0}{2}\int E^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{\varepsilon_0}{2}\int_0^\infty E^2 r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{4\pi\varepsilon_0}{2}\left[\int_0^R \left(\frac{k_e q}{R^3} r\right)^2 r^2 dr + \int_R^\infty \left(\frac{k_e q}{r^2}\right)^2 r^2 dr\right]$$

$$= k_e \frac{3}{5} \frac{q^2}{R}$$ as expected.

Problem 2.39: Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig. 2.49). At the center of each cavity a point charge is placed—call these charges q_a and q_b.

(a) Find the surface charge densities σ_a, σ_b, and σ_R. **(b)** What is the field outside the conductor?

(c) What is the field within each cavity? **(d)** What is the force on q_a and q_b?

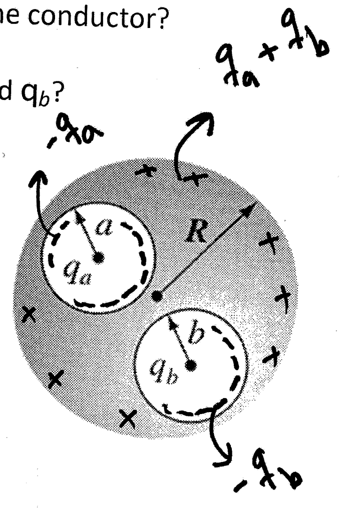
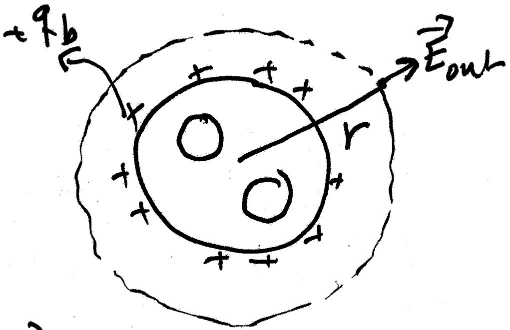


FIGURE 2.49

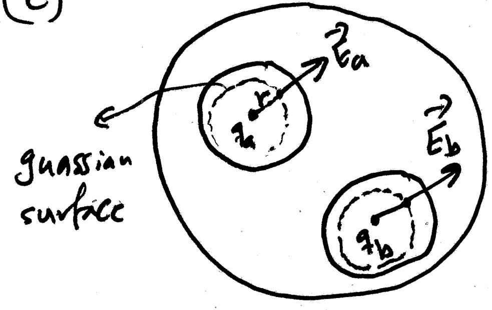
(a) $\sigma_a = -\dfrac{q_a}{4\pi a^2}$, $\sigma_b = -\dfrac{q_b}{4\pi b^2}$, $\sigma_R = \dfrac{q_a+q_b}{4\pi R^2}$

(b) $\vec{E}_{out} = ?$, take a gaussian surface with radius r

$$\vec{E}_{out} = k_e \frac{q_a+q_b}{r^2}\hat{r}$$



(c)



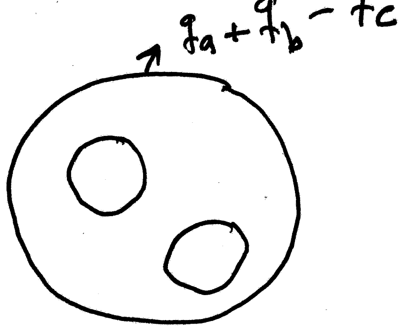
$$\vec{E}_a = k_e \frac{q_a}{r_a^2}\hat{r}_a$$

$$\vec{E}_b = k_e \frac{q_b}{r_b^2}\hat{r}_b$$

(d) $F_a = F_b =$ Zero as both charges q_a and q_b are screened from each other

(e) σ_a and σ_b are not affected $\Big\}$ q_c •

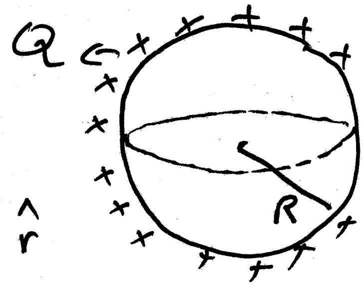
σ_R changes $= \dfrac{q_a+q_b-q_c}{4\pi R^2}$



and E_a and E_b are not affected, while $\vec{E}_{out}$ changes

Problem 2.42: A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

the force per unit area is

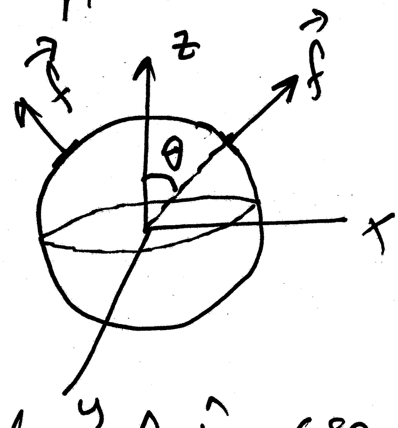


$\vec{f} = \sigma \vec{E}_{other}$; where $\sigma = \dfrac{Q}{4\pi R^2}$

and $\vec{E}_{other} = \dfrac{E_{out} + E_{in}}{2} = \dfrac{\sigma/\varepsilon_0 + 0}{2} = \dfrac{\sigma}{2\varepsilon_0}\hat{r}$

$\Rightarrow \vec{f} = \sigma \dfrac{\sigma}{2\varepsilon_0}\hat{r} = \dfrac{\sigma^2}{2\varepsilon_0}\hat{r} = \dfrac{Q^2}{32\varepsilon_0 \pi^2 R^4}\hat{r}$

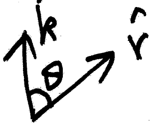
now due to spherical symmetry in the problem the cancel components of $\vec{f}$ projected on the x-y plane leaving only the z-component.



$f_z = \vec{f} \cdot \hat{k} = \dfrac{Q^2}{32\varepsilon_0 \pi^2 R^4}\cos\theta$

this is the force per unit area exerted by the southern hemisphere on the northern hemisphere. To get the total force exerted on the north hemisphere, we integrate over the north hemisphere $(0 < \theta < \pi/2)$, so we get

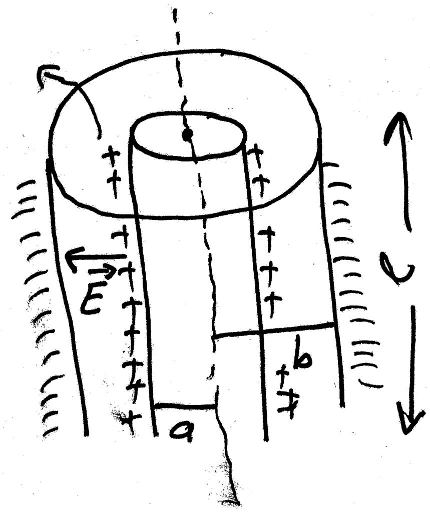
$\hat{r} \cdot \hat{k} = \cos\theta$



$$F_z = \int f_z \, da = \int \frac{Q^2}{32\varepsilon_0 \pi^2 R^4} \cos\theta \, R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q^2}{32\varepsilon_0 \pi^2 R^2} \underbrace{\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta}_{\frac{1}{2}\sin^2\theta \big|_0^{\pi/2} = \frac{1}{2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \frac{Q^2}{32\varepsilon_0 \pi R^2}$$

Problem 2.43: Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b (Fig. 2.53).



$$\Delta V = V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s}$$

where $d\vec{s} = ds\,\hat{s}$ and $\vec{E} = \dfrac{2k_e \lambda}{s}\,\hat{s}$

$$\Rightarrow \Delta V = -2k_e \lambda \int_a^b \frac{ds}{s}$$

$$= -2k_e \lambda \left[\ln s\right]_a^b = -2k_e \frac{Q}{l} \ln\left(\frac{b}{a}\right)$$

Now $$C = \frac{Q}{|\Delta V|} = \frac{\cancel{Q}}{2k_e \frac{\cancel{Q}}{l} \ln\left(\frac{b}{a}\right)} = \frac{l}{2k_e \ln(b/a)}$$

the capacitance per unit length is

$$\frac{C}{l} = \frac{1}{2k_e \ln(b/a)} = \frac{4\pi \varepsilon_0}{2k_e \ln(b/a)}$$

$$= \frac{2\pi \varepsilon_0}{\ln(b/a)}$$