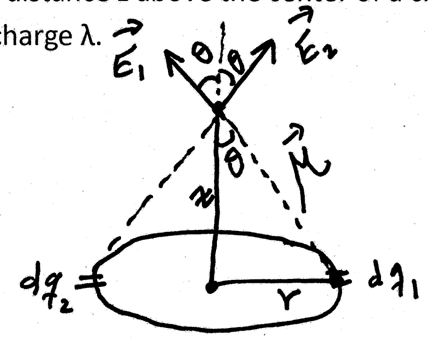
**Electromagnetic theory (1)**

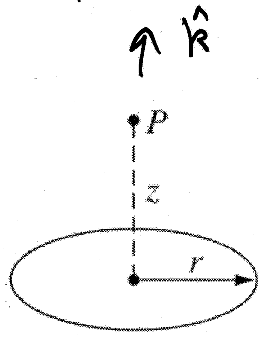
**Homework # 1 - Solution**

**Dr. Gassem Alzoubi**

**Problem 2.5**: Find the electric field a distance z above the center of a circular loop of radius r (Fig. 2.9) that carries a uniform line charge λ.

from symmetry, only vertical components survive (along z-direction)




$E_1 \cos\theta \uparrow \qquad \uparrow E_2 \cos\theta$

FIGURE 2.9

$-E_1 \sin\theta \leftarrow \qquad \rightarrow E_2 \sin\theta \quad ; \quad |E_1| = |E_2|$

Now $d\vec{E} = k_e \dfrac{dq}{\mu^2} \cos\theta \, \hat{k} \quad ; \quad \cos\theta = \dfrac{z}{|\vec{\mu}|} = \dfrac{z}{(z^2+r^2)^{1/2}}$

$= k_e \dfrac{dq \, z}{(r^2+z^2)^{3/2}} \hat{k} \qquad \mu^2 = z^2 + r^2$

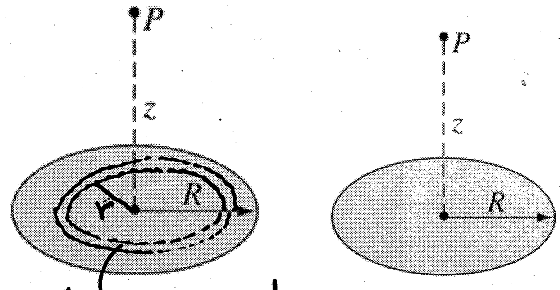
$$\vec{E} = \frac{k_e z}{(z^2+r^2)^{3/2}} \int dq \, \hat{k} = \frac{k_e z \, q}{(z^2+r^2)^{3/2}} \hat{k}$$

but $dq = \lambda \, ds \Rightarrow q = \lambda \int ds = \lambda s = \lambda (2\pi r)$

$$\Rightarrow \vec{E} = \frac{k_e \lambda (2\pi r) z}{(z^2+r^2)^{3/2}} \hat{k}$$

**Problem 2.6:** Find the electric field a distance z above the center of a flat circular disk of radius R (Fig. 2.10) that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

divide the disk into small rings. each ring has a radius r and carries a charge dq. The field generated by one ring is given by




$$d\vec{E} = k_e \frac{z}{(z^2+r^2)^{3/2}}\, dq\, \hat{k}, \quad \text{from last problem (2.5)} \quad dq = \sigma\, dA = \sigma\, 2\pi r\, dr$$

$$\Rightarrow \vec{E} = k_e \sigma z \pi \int_0^R \frac{2r\,dr}{(z^2+r^2)}\, \hat{k} \; ; \quad \text{let } u = z^2+r^2, \; du = 2r\,dr$$

$$\Rightarrow \vec{E} = k_e \sigma z \pi \int_{z^2}^{z^2+R^2} \frac{du}{u^{3/2}}\, \hat{k} = k_e \sigma z \pi \frac{u^{-1/2}}{-1/2} = -\frac{2 k_e \sigma z \pi}{\sqrt{u}}\Bigg|_{z^2}^{z^2+R^2} \hat{k}$$

$$= -2 k_e \sigma z \pi \left[\frac{1}{z^2+R^2} - \frac{1}{z}\right]\hat{k} = 2 k_e \sigma z \pi \left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{k}$$

if $R \to \infty \Rightarrow \vec{E} \to \frac{1}{4\pi\varepsilon_0} 2\sigma\pi\, \hat{k} = \frac{\sigma}{2\varepsilon_0}\hat{k}$ as expected for infinite sheet

Now if $z \gg R$, then expand $\frac{1}{\sqrt{z^2+R^2}}$ as

$$\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{z}\frac{1}{\left(1+\frac{R^2}{z^2}\right)^{1/2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)$$

where I used $(1+x)^n \approx 1+nx$ when $x \ll 1$, so

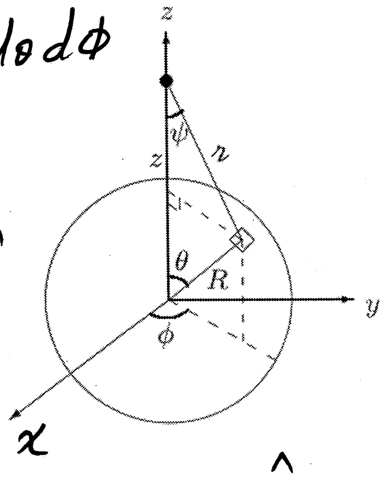
$$\vec{E} = 2 k_e \sigma z \pi \left[\cancel{\frac{1}{z}} - \cancel{\frac{1}{z}} + \frac{1}{2}\frac{R^2}{z^3}\right] = \frac{k_e \sigma z \pi R^2}{z^3} = k_e \frac{\sigma \pi R^2}{z^2}\hat{k}$$

but $Q = \sigma A = \sigma \pi R^2$

$$\Rightarrow \vec{E} = k_e \frac{Q}{z^2}\hat{k}$$ similar to point charge

**Problem 2.7:** Find the electric field a distance z from the center of a spherical surface of radius R that carries a uniform charge density σ. Treat the case z < R (inside) as well as z > R (outside). Express your answers in terms of the total charge q on the sphere. [Hint: Use the law of cosines to write r in terms of R and θ. Be sure to take the positive square root: $\sqrt{R^2+z^2-2Rz} = (R-z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]

$$\vec{E} = k_e \int \frac{dq}{|\vec{r}-\vec{r}'|}\,(\vec{r}-\vec{r}') \; ; \text{ where } dq = \sigma\, da' = \sigma R^2 \sin\theta\, d\theta\, d\phi$$




and $\vec{r} = z\hat{k}$ , $\vec{r}' = R\hat{r}$

$$\Rightarrow \vec{r}-\vec{r}' = z\hat{k} - R\hat{r} \; ;$$

but $\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$$\Rightarrow \vec{r}-\vec{r}' = z\hat{k} - R\sin\theta\cos\phi\,\hat{i} - R\sin\theta\sin\phi\,\hat{j} - R\cos\theta\,\hat{k}$$

$$= (z - R\cos\theta)\,\hat{k} - R\sin\theta\cos\phi\,\hat{i} - R\sin\theta\sin\phi\,\hat{j}$$

Now using the Law of cosine,

$$\mathscr{r} = |\vec{r}-\vec{r}'| = \sqrt{z^2+R^2-2Rz\cos\theta} = (z^2+R^2-2Rz\cos\theta)^{1/2}$$

$$\Rightarrow \vec{E} = k_e\sigma \int \frac{R^2\sin\theta\, d\theta\, d\phi}{(z^2+R^2-2Rz\cos\theta)^{3/2}}\,(\vec{r}-\vec{r}')$$

$$= k_e\sigma\left[ R^2 \iint \frac{(z-R\cos\theta)\sin\theta\, d\theta\, d\phi}{(z^2+R^2-2Rz\cos\theta)^{3/2}}\,\hat{k} - \underbrace{R\iint \frac{\sin^2\theta\cos\phi\, d\theta\, d\phi}{(z^2+R^2-2Rz\cos\theta)^{3/2}}}_{\text{zero since } \int_0^{2\pi}\cos\phi\, d\phi = 0}\,\hat{i} - \underbrace{R\iint \frac{\sin^2\theta\sin\phi\, d\theta\, d\phi}{(z^2+R^2-2Rz\cos\theta)^{3/2}}}_{\text{zero since } \int_0^{2\pi}\sin\phi\, d\phi = 0}\,\hat{j}\right]$$

$$\Rightarrow \vec{E} = k_e \sigma R^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\pi} \frac{(z - R\cos\theta)}{(z^2+R^2-2Rz\cos\theta)^{3/2}} \hat{k}$$

$$= 2\pi R^2 k_e \sigma \int_0^{\pi} \frac{(z - R\cos\theta)}{(z^2+R^2-2Rz\cos\theta)^{3/2}} \hat{k}$$

using integral calculator

$$= 2\pi R^2 \sigma k_e \left[ \frac{z+R}{z^2\sqrt{z^2+R^2+2Rz}} + \frac{z-R}{z^2\sqrt{z^2+R^2-2Rz}} \right]$$

$$= \frac{2\pi R^2 \sigma}{4\pi\varepsilon_0 z^2} \left[ \frac{z+R}{|z+R|} + \frac{z-R}{|z-R|} \right]$$

for $z > R \Rightarrow \vec{E} = \frac{R^2\sigma}{2\varepsilon_0 z^2} \left\{ \frac{z+R}{z+R} + \frac{z-R}{z-R} \right\} \hat{k}$

$$= \frac{R^2\sigma}{2\varepsilon_0 z^2} \{1+1\} = \frac{R^2\sigma}{\varepsilon_0 z^2} = \frac{4\pi R^2\sigma}{4\pi\varepsilon_0 z^2} \hat{k}$$

$$= k_e \frac{q}{z^2} \hat{k}$$ as a point charge

for $z < R$

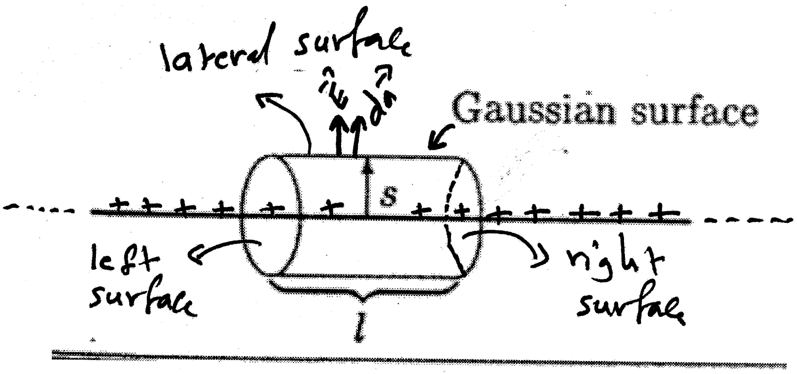
$$\vec{E} = \frac{R^2\sigma}{2\varepsilon_0 z^2} \left[ \frac{z+R}{|z+R|} + \frac{z-R}{|-(R-z)|} \right] = \frac{R^2\sigma}{2\varepsilon_0 z^2} \left\{ \frac{z+R}{z+R} + \frac{z-R}{R-z} \right\}$$

$$= \frac{R^2\sigma}{2\varepsilon_0 z^2} \left[ 1 - \frac{(R-z)}{R-z} \right] = \frac{R^2\sigma}{2\varepsilon_0 z^2} \{1-1\}$$

$$= \text{zero}$$

**Problem 2.13**: Find the electric field a distance s from an infinitely long straight wire that carries a uniform line charge λ

Field lines penetrate only the lateral surface, so




$$\oint_S \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\underbrace{\int_{left} \vec{E}\cdot\vec{da}}_{\substack{\text{zero} \\ \text{as } \vec{E}\perp\vec{da}}} + \underbrace{\int_{right} \vec{E}\cdot\vec{da}}_{\substack{\text{zero} \\ \vec{E}\perp\vec{da}}} + \int_{lateral} \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \int_{lateral} E\,da\cos\theta = \frac{Q_{enc}}{\varepsilon_0}$$ ; $\theta = 0$ and $E$ is constant over the surface

$$E\int da = \frac{Q_{enc}}{\varepsilon_0}$$

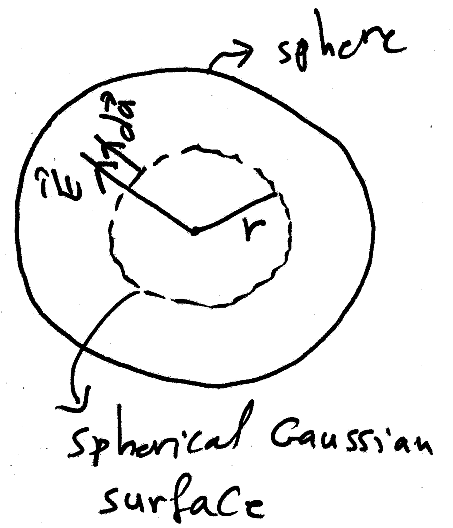
$$E(2\pi s)l = \frac{\lambda l}{\varepsilon_0}$$ ; as $Q_{enc} = \lambda l$

$$\Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 s} = \frac{2k_e\lambda}{s}$$ radially outward

i.e. $$\vec{E} = \frac{2k_e\lambda}{s}\hat{s}$$ ; $\hat{s}$ is unit vector in direction of $\vec{s}$

**Problem 2.14**: Find the electric field inside a sphere that carries a charge density proportional to the distance from the origin, ρ = kr, for some constant k. [Hint: This charge density is not uniform, and you must integrate to get the enclosed charge.]

consider a spherical surface of radius $r$ as a Gaussian surface, then




$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E\int da = \frac{Q_{enc}}{\varepsilon_0}$$

$$E\,A = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E(4\pi r^2) = \frac{Q_{enc}}{\varepsilon_0}$$

but $Q_{enc} = \int \rho\, d\tau = \int (kr)(\underbrace{r^2 \sin\theta\, dr\, d\theta\, d\phi}_{d\tau})$ → volume element in spherical coordin

$$= k\int r^3 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= k\frac{r^4}{4} \cdot 2 \cdot 2\pi = k\pi r^4$$

$$\Rightarrow E(4\cancel{\pi} r^2) = \frac{k\cancel{\pi} r^4}{\varepsilon_0} \Rightarrow E = \frac{k}{4\varepsilon_0} r^2 \times \frac{\pi}{\pi}$$

$$= \frac{k\pi r^2}{4\pi\varepsilon_0} \quad \text{radially outward}$$

i.e $\boxed{\vec{E} = \dfrac{k\pi r^2}{4\pi\varepsilon_0}\,\hat{r}}$

Note that $Q_{enc}$ can be also calculated by

$$Q_{enc} = \int \rho\, d\tau = \int \rho\, 4\pi r^2 dr = \int kr\, 4\pi r^2 dr = k\pi r^4$$

**Problem 2.16** A long coaxial cable (Fig. 2.26) carries a uniform *volume* charge density $\rho$ on the inner cylinder (radius $a$), and a uniform *surface* charge density on the outer cylindrical shell (radius $b$). This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field in each of the three regions: (i) inside the inner cylinder ($s < a$), (ii) between the cylinders ($a < s < b$), (iii) outside the cable ($s > b$). Plot $|\mathbf{E}|$ as a function of $s$.

$$|Q(+)| = |Q(-)|$$

on cable — on outer surface.

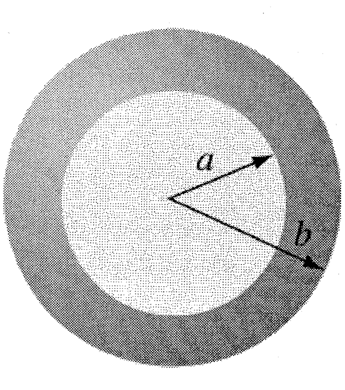



FIGURE 2.25

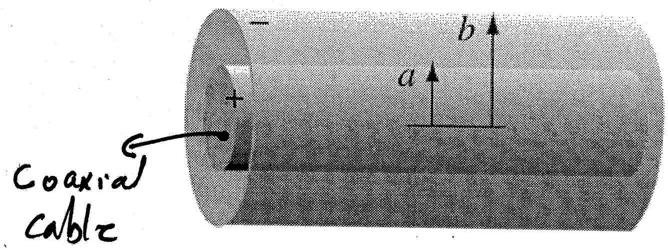
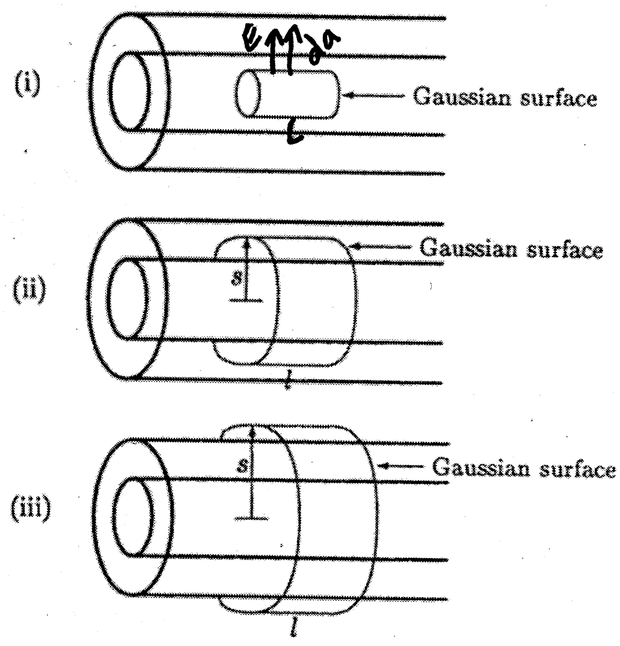



FIGURE 2.26




(i) $\oint \vec{E} \cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

$$E(2\pi s l) = \frac{1}{\varepsilon_0} \int \rho \, d\tau$$

$$E(2\pi s l) = \frac{1}{\varepsilon_0} \rho (\pi s^2 l)$$

$$\Rightarrow \vec{E} = \frac{\rho s}{2\varepsilon_0} \hat{s} \quad \text{radially outside}$$

(ii) $\oint \vec{E} \cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

$$E(2\pi s l) = \frac{1}{\varepsilon_0} \int \rho \, d\tau = \frac{1}{\varepsilon_0} \rho \, \pi a^2 l$$

$$\Rightarrow E = \frac{\rho a^2}{2\varepsilon_0 s} \quad \text{radially outside}, \quad \vec{E} = \frac{\rho a^2}{2\varepsilon_0 s} \hat{s}$$

(iii) $\oint \vec{E} \cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$ ; here $Q_{enc}$ = zero

$$\Rightarrow E = 0$$