The gradient obeys the following laws:

$$\operatorname{grad}(f+g) = \operatorname{grad} f + \operatorname{grad} g, \tag{3.10}$$

$$\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f; \tag{3.11}$$

that is, with the ∇ symbol,

$$\nabla(f+g) = \nabla f + \nabla g, \qquad \nabla(fg) = f\nabla g + g\nabla f. \tag{3.12}$$

These hold, provided grad f and grad g exist in the domain considered. The proofs are left for the problems.

If f is a constant c, (3.11) reduces to the simpler condition:

$$\operatorname{grad}(cg) = c \operatorname{grad} g \quad (c = \text{const}). \tag{3.13}$$

If the terms in z are dropped, the preceding discussion specializes at once to two dimensions. Thus for $f = f(x, y)$, one has

$$\begin{aligned} \operatorname{grad} f \equiv \nabla f &\equiv \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}, \\ \nabla &\equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}. \end{aligned} \tag{3.14}$$

PROBLEMS

1. Sketch the following vector fields:

a) $\mathbf{v} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$, **b)** $\mathbf{u} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$,

c) $\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$, **d)** $\mathbf{v} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$.

2. Sketch the level curves or surfaces of the following scalar fields:

a) $f = xy$, **b)** $f = x^2 + y^2 - z^2$, **c)** $f = e^{x+y-z}$.

3. Determine grad f for the scalar fields of Problem 2 and sketch several of the corresponding vectors.

4. Show that the gravitational field (3.2) is the gradient of the scalar

$$f = \frac{kMm}{r}.$$

5. Show that the force field (3.4) is the gradient of the scalar

$$f = \log \frac{\sqrt{(x-1)^2 + y^2}}{\sqrt{(x+1)^2 + y^2}}.$$

6. Prove (3.10) and (3.11).

7. Prove: If $f(x, y, z)$ is a composite function $F(u)$, where $u = g(x, y, z)$, then grad $f = F'(u)$ grad g.

8. Prove: grad $\frac{f}{g} = \frac{1}{g^2}$ [g grad $f - f$ grad g].

9. If $f = f(x_1, \dots, x_n)$, then the *Hessian matrix* of f is the matrix

$$H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$