The gradient obeys the following laws:

\[ \nabla (f + g) = \nabla f + \nabla g, \quad \text{(3.10)} \]
\[ \nabla (fg) = f \nabla g + g \nabla f ; \quad \text{(3.11)} \]

that is, with the \( \nabla \) symbol,

\[ \nabla (f + g) = \nabla f + \nabla g, \quad \nabla (fg) = f \nabla g + g \nabla f. \quad \text{(3.12)} \]

These hold, provided \( \nabla f \) and \( \nabla g \) exist in the domain considered. The proofs are left for the problems.

If \( f \) is a constant \( c \), (3.11) reduces to the simpler condition:

\[ \nabla (cg) = c \nabla g \quad (c = \text{const}). \quad \text{(3.13)} \]

If the terms in \( z \) are dropped, the preceding discussion specializes at once to two dimensions. Thus for \( f = f(x, y) \), one has

\[ \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}, \]
\[ \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j}. \quad \text{(3.14)} \]

**PROBLEMS**

1. Sketch the following vector fields:
   
   a) \( \mathbf{v} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}, \)
   
   b) \( \mathbf{u} = (x - y)\mathbf{i} + (x + y)\mathbf{j}, \)
   
   c) \( \mathbf{v} = -yi + x\mathbf{j} + k. \)
   
   d) \( \mathbf{v} = -xi - y\mathbf{j} - zk. \)

2. Sketch the level curves or surfaces of the following scalar fields:
   
   a) \( f = xy, \)
   
   b) \( f = x^2 + y^2 - z^2, \)
   
   c) \( f = e^{x+y-z}. \)

3. Determine \( \nabla f \) for the scalar fields of Problem 2 and sketch several of the corresponding vectors.

4. Show that the gravitational field (3.2) is the gradient of the scalar

   \[ f = \frac{km^m}{r}. \]

5. Show that the force field (3.4) is the gradient of the scalar

   \[ f = \log \frac{\sqrt{(x - 1)^2 + y^2}}{\sqrt{(x + 1)^2 + y^2}}. \]

6. Prove (3.10) and (3.11).

7. Prove: If \( f(x, y, z) \) is a composite function \( F(u) \), where \( u = g(x, y, z) \), then \( \nabla f = F'(u) \nabla g \).

8. Prove: \( \nabla \frac{f}{g} = \frac{1}{g} [g \nabla f - f \nabla g] \).

9. If \( f = f(x_1, \ldots, x_n) \), then the **Hessian matrix** of \( f \) is the matrix

   \[ H = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right). \]